Exact multi-restricted Schur polynomial correlators

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ABSTRACT: We derive a product rule satisfied by restricted Schur polynomials. We focus mostly on the case that the restricted Schur polynomial is built using two matrices, although our analysis easily extends to more than two matrices. This product rule allows us to compute exact multi-point correlation functions of restricted Schur polynomials, in the free field theory limit. As an example of the use of our formulas, we compute two point functions of certain single trace operators built using two matrices and three point functions of certain restricted Schur polynomials, exactly, in the free field theory limit. Our results suggest that gravitons become strongly coupled at sufficiently high energy, while the restricted Schur polynomials for totally antisymmetric representations remain weakly interacting at these energies. This is in perfect accord with the half-BPS (single matrix) results of hep-th/0512312. Finally, by studying the interaction of two restricted Schur polynomials we suggest a physical interpretation for the labels of the restricted Schur polynomial: the composite operator $\chi_{R,(r_n,r_m)}(Z,X)$ is constructed from the half BPS "partons" $\chi_{r_n}(Z)$ and $\chi_{r_m}(X)$.

Contents

1. Introduction

The AdS/CFT correspondence [1] has provided the possibility of studying quantum gravity non-perturbatively. A good example of this progress are the half-BPS type IIB geometries in asymptotically AdS spaces, constructed by Lin, Lunin and Maldacena [2] and their connection with the semiclassical states of a free fermi system [3, 4]. It would be fascinating to extend these studies beyond the half-BPS sector. The present work is directed at this goal.

In the boundary super Yang-Mills theory, the operators relevant for the half-BPS LLM geometries are constructed using a single free complex Higgs field Z. The dynamics of a single free matrix is captured by N free fermions in a harmonic potential [3–7]. At large N this single matrix dynamics has a semiclassical description in terms of droplets of fermi fluid in phase space. Remarkably, LLM showed that there is a similar structure in the classical geometries in the bulk.

To go beyond the half-BPS sector, one needs to study multi-matrix dynamics. In general, this is a formidable problem. There has however, been some recent progress: three independent bases for general multi-matrix models have been identified. For a review of these developments and the work leading up to them, see [8]. The basis described in [9] builds operators with definite flavor quantum numbers; we call this basis the flavor basis. The basis of [10] uses the Brauer algebra to build correlators involving Z and $Z^\dagger$; we call this basis the Brauer basis. The Brauer basis seems to be the most natural for exploring brane/anti-brane systems. The basis of [11] most directly allows one to consider open string excitations [12–16] of the operator; we call this the restricted Schur basis. These three bases do not coincide and a detailed link between them is an interesting open problem. All three bases diagonalize the two point functions in the free field theory limit.

Completeness of the flavor basis was convincingly demonstrated [9] by arguing that the number of such operators matches the number of gauge invariant operators that can be constructed [17], at both infinite and finite N. The flavor basis also gives a group theoretic way to approach higher point functions (see [9] where three and higher point functions are obtained) and to obtain factorization equations which can be used to build a probability interpretation [18]. By exploiting supergroups [9] have also explained how to include fermions in addition to the Higgs fields. Finally, the one loop correction to these two points functions has been considered in [19].

Arguments for the completeness of the restricted Schur basis in [11] demonstrated the number of restricted Schur polynomials matches the number of gauge invariant operators that can be constructed [17], again at both infinite and finite N, for a number of examples. A proof of this matching for infinite N is known [20]. In this article we will compute exact three and higher point correlation functions of restricted Schur polynomials, in the free field theory limit.

Our approach to the computation of multipoint functions has a simple algebraic description: We start by deriving a product rule on the space of restricted Schur polynomials: the product of any two restricted Schur polynomials can be expressed as a linear combination of restricted Schur polynomials. Applying the rule $(n-1)$ times we can collapse the product of n restricted Schur polynomials to a linear combination of restricted Schur polynomials. In this way, an arbitrary multipoint function can be reduced to a linear combination of two point functions - something that we know how to compute. These multipoint functions contain dynamical information about the theory, and hence they should reveal interesting physics. As an example, studying the Yang-Mills theory at finite N probes truly quantum mechanical aspects of the bulk gravity. At finite N only fluctuations with a low enough excitation energy can be interpreted as gravity modes propagating in the bulk. It is appropriate, at low energies, to use the graviton degrees of freedom to set up a perturbative description. At high energies, the description will employ more fundamental microscopic degrees of freedom. Since

$$\frac{R}{l_p} \sim N^{\frac{1}{4}},$$

with R the AdS radius and l_p the ten-dimensional Planck scale, one might expect a break-

down of the low energy description at energies $\sim N^{\frac{1}{4}}$ [22]. Exact finite N multi-point correlation functions were used in [22] to explore this expectation. Using an operator bosonization of a finite number of nonrelativistic fermions [21], exact computations of three-point correlators show that perturbation theory only breaks down at the scale $N^{\frac{1}{2}}$ [22]. There are at least two other arguments for this scale: in [23] the groundstate wavefunction [24] for a scalar field on AdS_5 arising from a mode with large angular momentum l was studied. The size of this wavefunction for an LLM graviton decreases with energy; at an energy scale of $\sim N^{\frac{1}{2}}$ it becomes of order the Plank scale. An argument for this scale can also be made directly in the field theory: for operators constructed with a small number of Higgs fields, operators with the same number of fields but with a different number of traces are orthogonal at large N. Identifying the number of traces in an operator with particle number, the supergravity Fock space structure emerges. However, when the number of fields are of order $N^{\frac{1}{2}}$, operators with a different number of traces are no longer orthogonal [25, 26] and "trace number" is just not a good quantum number. To go beyond these energies, one needs to employ a new set of degrees of freedom that are weakly coupled and hence provide a more meaningful description of the bulk physics than gravitons. In [22] it was argued that these new degrees of freedom are the giant gravitons. The point is that even at high energies, giant gravitons remain weakly interacting. The relevant correlator computations are in the half-BPS sector. By using our product rule we can compute giant graviton correlators that go beyond the half-BPS sector. Our results are consistent with those of [22] and we find that even in this more general setting giant gravitons continue to furnish suitable high energy degrees of freedom.

Schur polynomials built from a single matrix have, by now, a clear interpretation in the dual gravitational theory. Developing an interpretation for restricted Schur polynomials in an important problem. Our methods allow a study of the interaction of two restricted Schur polynomials, which should shed light on this issue. This is indeed the case: we suggest a physical interpretation for the labels of the restricted Schur polynomial: the composite operator $\chi_{R,(r_n,r_m)}(Z,X)$ is constructed from the half BPS "partons" $\chi_{r_n}(Z)$ and $\chi_{r_m}(X)$. Evidence for this proposal comes from considering composite operators in which the constituent partons are weakly interacting: the interaction of two such composites is largely determined by the interactions of the partons.

In the remainder of this introduction we will describe the organization of this paper. Section 2 describes the product rule satisfied by the restricted Schur polynomials. We focus on polynomials built using two complex Higgs fields, X and Z. The analogous product rule for the Schur polynomials is determined by the Littlewood-Richardson numbers. For this reason, we call the numbers that enter our rule *restricted Littlewood-Richardson numbers*. In section 3 we develop techniques that can be used to evaluate the restricted Littlewood-Richardson numbers. Essentially, we perform a change of basis so that in the new basis we can compute the restricted characters using the strand diagram techniques developed in [16]. These methods are not very efficient and one is not able, in general, to effectively deal with restricted Schur polynomials which have both a large number of X fields and a large number of Z fields. There are however special cases for which we can obtain explicit

results. These results are used in section 4 to describe some aspects of the fully quantum mechanical bulk gravity. In particular, we provide evidence that the giant gravitons (restricted Schur polynomials for the totally antisymmetric representations) furnish suitable high energy degrees of freedom, extending the half-BPS studies of [22]. In section 5 we present the evidence for our proposal that the composite operator $\chi_{R,(r_n,r_m)}(Z,X)$ is constructed from the half BPS "partons" $\chi_{r_n}(Z)$ and $\chi_{r_m}(X)$. In section 6 we discuss our results and outline some open problems.

2. A product rule for restricted Schur polynomials

The Schur polynomial $\chi_R(U)$ gives the character of $U \in \mathrm{SU}(N)$ in the $\mathrm{SU}(N)$ irreducible representation labeled by Young diagram R. The representation obtained by taking the direct product of two irreducible $\mathrm{SU}(N)$ representations R_1 and R_2 is in general reducible. The number of times that irreducible representation T appears is given by the Littlewood-Richardson number $f_{R_1 R_2;T}$. Using the fact that the Schur polynomials compute characters, it is clear that

$$\chi_{R_1}(Z)\chi_{R_2}(Z) = \sum_T f_{R_1 R_2;T}\chi_T(Z).$$

Using this product rule, it is easy to compute multipoint functions in terms of two point functions. For example, for three point functions we have

$$\langle \chi_{R_1}(Z)\chi_{R_2}(Z)\chi_S(Z)^\dagger \rangle = \sum_T f_{R_1 R_2;T}\langle \chi_T(Z)\chi_S(Z)^\dagger \rangle,$$

and for four point functions

$$\begin{aligned}\langle \chi_{R_1}(Z)\chi_{R_2}(Z)\chi_{R_3}(Z)\chi_S(Z)^\dagger \rangle &= \sum_T f_{R_1 R_2;T}\langle \chi_T(Z)\chi_{R_3}(Z)\chi_S(Z)^\dagger \rangle \\ &= \sum_T \sum_U f_{R_1 R_2;T}f_{R_3 T;U}\langle \chi_U(Z)\chi_S(Z)^\dagger \rangle.\end{aligned}$$

These results are in perfect agreement with the exact computations of [3, 27]. It is also clear that we can compute n-point functions knowing only the product rule and the two point functions. In this section we will argue that the restricted Schur polynomials themselves satisfy a simple product rule.

2.1 Dual characters

To begin, we review appendix I of [14]. If two permutations σ, τ satisfy

$$\mathrm{Tr}\,(\sigma Z^{\otimes n} \otimes X^{\otimes m}) = \mathrm{Tr}\,(\tau Z^{\otimes n} \otimes X^{\otimes m}),$$

we say they are *restricted conjugate*. Restricted conjugate is an equivalence relation. Clearly two elements σ and τ are restricted conjugate if they satisfy

$$\sigma = \mu^{-1}\tau\mu,$$

for some $\mu \in S_n \times S_m$. Denote the number of restricted conjugate classes by $N(n,m)$ and let $n^\sigma(n,m)$ denote the number of elements in the restricted conjugate class with representative σ. $N(n,m)$ is also equal to the number of restricted Schur polynomials $\chi_{R,(r_n,r_m)}$ with R an irreducible representation of S_{n+m} and (r_n,r_m) an irreducible representation of $S_n \times S_m$. The equality between the number of restricted conjugate classes $N(n,m)$ and the total number of labels $R,(r_n,r_m)$ generalizes the familiar equality for the symmetric group of the number of conjugacy classes and the number of irreducible representations. Introduce the matrix

$$(M^{-1})_{\sigma\tau} = \sum_{(R,(r_n,r_m))} \chi_{R,(r_n,r_m)}(\sigma)\chi_{R,(r_n,r_m)}(\tau), \tag{2.1}$$

where the sum on the right hand side runs over all possible labels $(R,(r_n,r_m))$. The restricted character is defined by [14]

$$\chi_{R,(r_n,r_m)}(\sigma) = \text{Tr}_{(r_n,r_m)}\left(\Gamma_R(\sigma)\right).$$

Define the *dual restricted character* (which we denote by a superscript) by

$$\chi^{R,(r_n,r_m)}(\sigma) = \frac{n!m!}{n^\sigma(n,m)} \sum_{[\tau]_r} M_{\sigma\tau}\chi_{R,(r_n,r_m)}(\tau).$$

The indices σ and τ that appear in (2.1) run over the restricted conjugacy classes. Denote the restricted conjugacy class with representative ψ by $[\psi]_r$. It is clear that

$$\sum_{R,(r_n,r_m)} \chi^{R,(r_n,r_m)}(\sigma)\chi_{R,(r_n,r_m)}(\rho) = n!m!\delta([\sigma]_r[\rho]_r). \tag{2.2}$$

In appendix A we derive a formula for the dual restricted character in terms of the restricted character. To correctly state this formula, we need to spell out both the row and the column indices that are summed in the trace. For an "on the diagonal block" trace, the column and row indices that are summed come from the same carrier space and we are summing diagonal elements of the matrix; for an "off the diagonal block" trace, the column and row indices that are traced come from distinct carrier spaces so that we are summing off diagonal elements of the matrix.[1] Indicating both row and column indices of the restricted trace, the dual character is given by

$$\chi^{R,((r_n,r_m),(s_n,s_m))}(\tau) = \frac{d_R n! m!}{d_{r_n} d_{r_m}(n+m)!}\chi_{R,((s_n,s_m),(r_n,r_m))}(\tau). \tag{2.3}$$

2.2 Product rule

We will now define what we call *restricted Littlewood-Richardson numbers*. The restricted Littlewood-Richardson numbers determine the product rule for restricted Schur polynomials in exactly the same way that the Littlewood-Richardson numbers determine the product rule for Schur polynomials. Let R_1 be an irreducible representation of $S_{n_1+m_1}$

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[1]See [13, 14] for more details.

and let (r_{n_1}, r_{m_1}) be an irreducible representation of $S_{n_1} \times S_{m_1}$. Let R_2 be an irreducible representation of $S_{n_2+m_2}$ and let (r_{n_2}, r_{m_2}) be an irreducible representation of $S_{n_2} \times S_{m_2}$. Finally, let R_{1+2} be an irreducible representation of $S_{n_1+n_2+m_1+m_2}$ and let $(r_{n_{1+2}}, r_{m_{1+2}})$ be an irreducible representation of $S_{n_1+n_2} \times S_{m_1+m_2}$. The restricted Littlewood-Richardson numbers are defined by

$$f^{R_{1+2},(r_{n_{1+2}},r_{m_{1+2}})}_{R_1,(r_{n_1},r_{m_1})\,R_2,(r_{n_2},r_{m_2})} = \frac{1}{n_1!n_2!m_1!m_2!} \tag{2.4}$$
$$\times \sum_{\sigma_1 \in S_{n_1+m_1}} \sum_{\sigma_2 \in S_{n_2+m_2}} \chi_{R_1,(r_{n_1},r_{m_1})}(\sigma_1) \chi_{R_2,(r_{n_2},r_{m_2})}(\sigma_2) \chi^{R_{1+2},(r_{n_{1+2}},r_{m_{1+2}})}(\sigma_1 \circ \sigma_2).$$

To streamline the notation, from now on we will replace the composite label $R_i, (r_{n_i}, r_{m_i})$ simply by $\{i\}$ and we define $n_{12} \equiv n_1 + n_2$, $m_{12} \equiv m_1 + m_2$. With the new streamlined notation we write

$$f^{\{1+2\}}_{\{1\}\{2\}} = \frac{1}{n_1!n_2!m_1!m_2!} \sum_{\sigma_1 \in S_{n_1+m_1}} \sum_{\sigma_2 \in S_{n_2+m_2}} \chi_{\{1\}}(\sigma_1) \chi_{\{2\}}(\sigma_2) \chi^{\{1+2\}}(\sigma_1 \circ \sigma_2).$$

The restricted Schur product rule says

$$\chi_{\{1\}}(Z,X)\chi_{\{2\}}(Z,X) = \sum_{\{1+2\}} f^{\{1+2\}}_{\{1\}\{2\}} \chi_{\{1+2\}}(Z,X). \tag{2.5}$$

A few comments are in order. The restricted Schur polynomial $\chi_{\{1+2\}}(Z,X)$ is given by

$$\chi_{\{1+2\}}(Z,X) = \frac{1}{n_{12}!m_{12}!} \sum_{\rho \in S_{n_{12}+m_{12}}} \text{Tr}_{(r_{n_{1+2}},r_{m_{1+2}})} \left(\Gamma_{R_{1+2}}(\rho)\right) \text{Tr}\left(\rho Z^{\otimes n_{12}} \otimes X^{\otimes m_{12}}\right).$$

$(r_{n_{1+2}}, r_{m_{1+2}})$ is an irreducible representation of the $S_{n_{12}} \times S_{m_{12}}$ subgroup which permutes indices of the Zs amongst each other and the indices of the Xs amongst each other. The representation R_1 is an irreducible representation of the $S_{n_1+m_1}$ subgroup that acts on the first n_1 Zs and the first m_1 Xs; the representation R_2 is an irreducible representation of the $S_{n_2+m_2}$ subgroup that acts on the remaining n_2 Zs and m_2 Xs. To demonstrate the restricted Schur product rule, consider

$$\sum_{\{1+2\}} f^{\{1+2\}}_{\{1\}\{2\}} \chi_{\{1+2\}}(Z,X) = \frac{1}{n_1!n_2!m_1!m_2!} \tag{2.6}$$
$$\times \sum_{\{1+2\}} \sum_{\sigma_1 \in S_{n_1+m_1}} \sum_{\sigma_2 \in S_{n_2+m_2}} \chi_{\{1\}}(\sigma_1) \chi_{\{2\}}(\sigma_2) \chi^{\{1+2\}}(\sigma_1 \circ \sigma_2)$$
$$\times \frac{1}{n_{12}!m_{12}!} \sum_{\rho \in S_{n_{12}+m_{12}}} \text{Tr}_{(r_{n_{1+2}},r_{m_{1+2}})} \left(\Gamma_{R_{1+2}}(\rho)\right) \text{Tr}\left(\rho Z^{\otimes n_1+n_2} \otimes X^{\otimes m_1+m_2}\right).$$

After using (2.2) we obtain

$$\sum_{\{1+2\}} f^{\{1+2\}}_{\{1\}\{2\}} \chi_{\{1+2\}}(Z,X) = \frac{1}{n_1!n_2!m_1!m_2!} \sum_{\sigma_1 \in S_{n_1+m_1}} \sum_{\sigma_2 \in S_{n_2+m_2}} \sum_{\rho \in S_{n_{12}+m_{12}}} \chi_{\{1\}}(\sigma_1) \chi_{\{2\}}(\sigma_2) \times$$
$$\times \delta\left(\left[\sigma_1 \circ \sigma_2\right]_r \left[\rho\right]_r\right) \text{Tr}\left(\rho Z^{\otimes n_{12}} \otimes X^{\otimes m_{12}}\right)$$

$$= \frac{1}{n_1!n_2!m_1!m_2!} \sum_{\sigma_1 \in S_{n_1+m_1}} \sum_{\sigma_2 \in S_{n_2+m_2}} \chi_{\{1\}}(\sigma_1)\chi_{\{2\}}(\sigma_2)\mathrm{Tr}\left(\sigma_1 \circ \sigma_2 Z^{\otimes n_{12}} \otimes X^{\otimes m_{12}}\right)$$

$$= \frac{1}{n_1!m_1!} \sum_{\sigma_1 \in S_{n_1+m_1}} \chi_{\{1\}}(\sigma_1)\mathrm{Tr}\left(\sigma_1 Z^{\otimes n_1} \otimes X^{\otimes m_1}\right) \frac{1}{n_2!m_2!} \sum_{\sigma_2 \in S_{n_2+m_2}} \chi_{\{2\}}(\sigma_2)\mathrm{Tr}\left(\sigma_2 Z^{\otimes n_2} \otimes X^{\otimes m_2}\right)$$

$$= \chi_{\{1\}}(Z,X)\chi_{\{2\}}(Z,X), \tag{2.7}$$

which proves the product rule.

Using the explicit formula for the dual character (2.3) leads to the formula

$$f_{R_1,(r_{n_1},r_{m_1})\,R_2,(r_{n_2},r_{m_2})}^{R_{1+2},(r_{n_{1+2}},r_{m_{1+2}})(s_{n_{1+2}},s_{m_{1+2}})} = \frac{n_{12}!m_{12}!}{n_1!n_2!m_1!m_2!(n_{12}+m_{12})!}$$

$$\times \frac{d_{R_{1+2}}}{d_{r_{n_{1+2}}}d_{r_{m_{1+2}}}} \sum_{\sigma_1 \in S_{n_1+m_1}} \sum_{\sigma_2 \in S_{n_2+m_2}} \chi_{R_1,(r_{n_1},r_{m_1})}(\sigma_1)\chi_{R_2,(r_{n_2},r_{m_2})}(\sigma_2)$$

$$\times \chi_{R_{1+2},(s_{n_{1+2}},s_{m_{1+2}})(r_{n_{1+2}},r_{m_{1+2}})}(\sigma_1 \circ \sigma_2) \tag{2.8}$$

for the restricted Littlewood-Richardson numbers.

3. Computation of the restricted Littlewood-Richardson numbers

In this section we will develop rules that will allow us to compute the restricted characters needed to evaluate the restricted Littlewood Richardson numbers. A general diagrammatic method, *strand diagrams*, to compute restricted characters in the case that the polynomial is built from a single matrix has been developed in [16]. In this section we would like to develop methods that are powerful enough to allow the computation of restricted characters for polynomials built out of both Z and X. It is enough to compute the characters of two cycles, since any element can be decomposed into a product of two cycles. It is precisely this fact that was exploited to build the strand diagrams of [16]. In the next two sections the character of arbitrary two cycles for an on the diagonal block restriction and then an off the diagonal block restriction are computed. Finally, some example computations of restricted Littlewood-Richardson coefficients are discussed.

3.1 On the diagonal restricted characters of two cycles

We need two pieces of information: first, we will introduce three Casimirs that will be particularly useful. Second, we will argue that all characters for two cycles which do not belong to $S_n \times S_m$ are equal. Taken together, these two facts will allow us to compute the restricted character of an arbitrary two cycle.

Let the first n indices be associated with the Z matrices and the next m indices be associated with the X fields. Greek indices run over $\alpha = 1, 2, \dots, m+n$. Indices from the start of the alphabet run over $a = 1, 2, \dots, n$. Indices from the middle of the alphabet run over $i = n+1, n+2, \dots, n+m$. The operator

$$\hat{O}_{n+m} = \sum_{\alpha<\beta=1}^{n+m} (\alpha\beta)$$

is a Casimir of S_{n+m}. When acting on an irreducible representation described by a Young Diagram R with r_i boxes in row i and c_j boxes in column j it takes the value

$$\hat{O}_{n+m}|R\rangle = \Big[\sum_i r_i(r_i - 1) - \sum_j c_j(c_j - 1) \Big]|R\rangle.$$

The operators

$$\hat{O}_n = \sum_{a<b=1}^{n} (ab), \qquad \hat{O}_m = \sum_{i<j=n+1}^{n+m} (ij),$$

are Casimirs of $S_n \times S_m$. Consider the $S_n \times S_m$ irreducible representation R, described by Young diagrams r_n (for the Zs) and r_m (for the Xs). r_n has $r_{1,i}$ boxes in row i and $c_{1,j}$ boxes in column j; r_m has $r_{2,i}$ boxes in row i and $c_{2,j}$ boxes in column j. These Casimirs take the values

$$\hat{O}_n|(r_n, r_m)\rangle = \Big[\sum_i r_{1,i}(r_{1,i} - 1) - \sum_j c_{1,j}(c_{1,j} - 1) \Big]|(r_n, r_m)\rangle,$$

$$\hat{O}_m|(r_n, r_m)\rangle = \Big[\sum_i r_{2,i}(r_{2,i} - 1) - \sum_j c_{2,j}(c_{2,j} - 1) \Big]|(r_n, r_m)\rangle.$$

If a cycle belongs to $S_n \times S_m$ it has the form (ab) or (ij). In this case

$$\mathrm{Tr}_{(r_n,r_m)}((ab)) = \mathrm{Tr}_{r_n}((ab))d_{r_m}, \quad \text{and} \quad \mathrm{Tr}_{(r_n,r_m)}((ij)) = \mathrm{Tr}_{r_m}((ij))d_{r_n}.$$

We can calculate $\mathrm{Tr}_{r_n}((ab))$ and $\mathrm{Tr}_{r_m}((ij))$ using the results of [16]. We will now argue that all characters for two cycles which do not belong to $S_n \times S_m$ are equal. Any two such cycles can be related as

$$\begin{aligned}
\Gamma_R((aj)) &= \Gamma_R((ab))\Gamma_R((jl))\Gamma_R((bl))\Gamma_R((jl))\Gamma_R((ab)) \\
&= \Gamma_R((ab))^{-1}\Gamma_R((jl))^{-1}\Gamma_R((bl))\Gamma_R((jl))\Gamma_R((ab)).
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathrm{Tr}_{(r_n,r_m)}\Big(\Gamma_R((aj))\Big) &= \mathrm{Tr}_{(r_n,r_m)}\Big(\Gamma_R((ab))^{-1}\Gamma_R((jl))^{-1}\Gamma_R((bl))\Gamma_R((jl))\Gamma_R((ab))\Big) \\
&= \mathrm{Tr}_{(r_n,r_m)}\Big(\Gamma_{R_\gamma}((ab))^{-1}\Gamma_{R_\gamma}((jl))^{-1}\Gamma_R((bl))\Gamma_{R_\gamma}((jl))\Gamma_{R_\gamma}((ab))\Big) \\
&= \mathrm{Tr}_{(r_n,r_m)}\Big(\Gamma_R((bl))\Big).
\end{aligned}$$

Clearly, all characters for two cycles which do not belong to $S_n \times S_m$ are equal.

Taken together, these two facts imply that (the cycle (aj) does not belong to $S_n \times S_m$)

$$\begin{aligned}
\mathrm{Tr}_{(r_n,r_m)}\Big(\Gamma_R((aj))\Big) &= \frac{1}{nm} \sum_{i=n+1}^{n+m} \sum_{a=1}^{n} \mathrm{Tr}_{(r_n,r_m)}\Big(\Gamma_R((ia))\Big) \\
&= \frac{1}{nm} \mathrm{Tr}_{(r_n,r_m)}(\hat{O}_{n+m} - \hat{O}_n - \hat{O}_m). \qquad (3.1)
\end{aligned}$$

The eigenvalue of the Casimir operator $\hat{O}_{n+m} - \hat{O}_n - \hat{O}_m$ can be written as the sum of the weights of R minus the sum of the weights of r_n minus the sum of the weights of r_m. This

observation allows us to derive much simpler formulas for the case that r_m say, contains only a few boxes. Indeed, we can imagine that (r_n, r_m) was formed by peeling boxes off R to leave r_n and then combining the peeled boxes to form r_m. In this case, the eigenvalue of $\hat{O}_{n+m} - \hat{O}_n - \hat{O}_m$ is given by the sum of weights of the boxes peeled off R minus the sum of the weights of the boxes in r_m. We will now give the simplified versions of (3.1) for the cases that $m = 1, 2$ or 3.

Restricted character for $m = 1$. In the following formula, R is an irreducible representation of S_{n+1} and r_n is an irreducible representation of S_n of dimension d_{r_n}. A single box must be removed from R to obtain r_n. Denote the weight of the box that must be removed by c_1. The simplified character formula is

$$\text{Tr}_{(r_n, \square)}\Big(\Gamma_R\big((aj)\big)\Big) = \frac{c_1 - N}{n} d_{r_n}.$$

Restricted character for $m = 2$. In the following formula, R is an irreducible representation of S_{n+2} and r_n is an irreducible representation of S_n of dimension d_{r_n}. Two boxes must be removed from R to obtain r_n. Denote the weights of the boxes that must be removed by c_1 and c_2. The simplified character formulas are

$$\text{Tr}_{(r_n, \square\square)}\Big(\Gamma_R\big((aj)\big)\Big) = \frac{c_1 + c_2 - 2N - 1}{2n} d_{r_n},$$
$$\text{Tr}_{(r_n, \square)}\Big(\Gamma_R\big((aj)\big)\Big) = \frac{c_1 + c_2 - 2N + 1}{2n} d_{r_n}.$$

Restricted character for $m = 3$. In the following formula, R is an irreducible representation of S_{n+3} and r_n is an irreducible representation of S_n of dimension d_{r_n}. Three boxes must be removed from R to obtain r_n. Denote the weights of the boxes that must be removed by c_1, c_2 and c_3. The simplified character formulas are

$$\text{Tr}_{(r_n, \square\square\square)}\Big(\Gamma_R\big((aj)\big)\Big) = \frac{c_1 + c_2 + c_3 - 3N - 3}{3n} d_{r_n},$$
$$\text{Tr}_{(r_n, \square)}\Big(\Gamma_R\big((aj)\big)\Big) = 2\frac{c_1 + c_2 + c_3 - 3N}{3n} d_{r_n},$$
$$\text{Tr}_{(r_n, \square)}\Big(\Gamma_R\big((aj)\big)\Big) = \frac{c_1 + c_2 + c_3 - 3N + 3}{3n} d_{r_n}.$$

3.2 Off the diagonal restricted characters of two cycles

Before we can compute generic restricted characters, we need to compute traces over off the diagonal blocks: $\text{Tr}_{(r_n, r_m),(s_n, s_m)}\Big(\Gamma_R\big((\alpha\beta)\big)\Big)$, where (r_n, r_m) and (s_n, s_m) are distinct $S_n \times S_m$ representations. This character clearly vanishes if $(\alpha\beta)$ belongs to the $S_n \times S_m$ subgroup. What about the mn two cycles that do not belong to $S_n \times S_m$? For concreteness, consider the computation of $\text{Tr}_{(r_n, r_m)(s_n, s_m)}\Big(\Gamma_R\big((ai)\big)\Big)$. Let $(r_n, r_m)'_I$ denote the complete set of irreducible representations of the $S_{n-1} \times S_{m-1}$ subgroup subduced by (r_n, r_m), that is $\oplus_I (r_n, r_m)'_I = (r_n, r_m)$. The $S_{n-1} \times S_{m-1}$ subgroup of interest is obtained by keeping all elements of $S_n \times S_m$ that hold indices a and i fixed. The representations (r_n, r_m)

and (s_n, s_m) have the same shape[2] so that we can establish a bijective map between their bases. We will assume that this bijective map is the identity, which we can always arrange by a suitable choice of basis. This choice of basis ensures that when we subduce to the $S_{n-1} \times S_{m-1}$ subgroup we have

$$\text{Tr}_{(r_n, r_m)(s_n, s_m)}\Big(\Gamma_R\big((ai)\big)\Big) = \sum_I \text{Tr}_{(r_n, r_m)'_I(s_n, s_m)'_I}\Big(\Gamma_R\big((ai)\big)\Big).$$

We will now provide further insight into this formula. It is straight forward to prove that

$$\langle (r_n, r_m)'_I, i | \Gamma_R\big((ai)\big)\Big) | (s_n, s_m)'_I \rangle$$

vanishes unless $(r_n, r_m)'_I$ and $(s_n, s_m)'_I$ have the same shape. Introduce the Casimirs

$$\hat{O}_{n-1} = \sum_{c<b=1\neq a}^{n} (cb), \qquad \hat{O}_{m-1} = \sum_{k<j=n+1\neq i}^{n+m} (kj).$$

Denote the eigenvalue of Casimir $\hat{O}_{n-1}$ for the $(r_n, r_m)'_I$ representation by $\lambda_r^{(n-1)}$ and the eigenvalue of Casimir $\hat{O}_{m-1}$ for the $(r_n, r_m)'_I$ representation by $\lambda_r^{(m-1)}$. Clearly, we have

$$\begin{aligned}
\lambda_r^{(n-1)}\langle (r_n, r_m)'_I, i | \Gamma_R\big((ai)\big) | (s_n, s_m)'_I \rangle &= \langle (r_n, r_m)'_I, i | \hat{O}_{n-1}\Gamma_R\big((ai)\big) | (s_n, s_m)'_I \rangle \\
&= \langle (r_n, r_m)'_I, i | \Gamma_R\big((ai)\big)\hat{O}_{n-1} | (s_n, s_m)'_I \rangle \\
&= \lambda_s^{(n-1)}\langle (r_n, r_m)'_I, i | \Gamma_R\big((ai)\big) | (s_n, s_m)'_I \rangle, \\
\lambda_r^{(m-1)}\langle (r_n, r_m)'_I, i | \Gamma_R\big((ai)\big) | (s_n, s_m)'_I \rangle &= \langle (r_n, r_m)'_I, i | \hat{O}_{m-1}\Gamma_R\big((ai)\big) | (s_n, s_m)'_I \rangle \\
&= \langle (r_n, r_m)'_I, i | \Gamma_R\big((ai)\big)\hat{O}_{m-1} | (s_n, s_m)'_I \rangle \\
&= \lambda_s^{(m-1)}\langle (r_n, r_m)'_I, i | \Gamma_R\big((ai)\big) | (s_n, s_m)'_I \rangle,
\end{aligned}$$

which implies that

$$\langle (r_n, r_m)'_I, i | \Gamma_R\big((ai)\big) | (s_n, s_m)'_I \rangle \propto \delta_{\lambda_r^{(n-1)} \lambda_s^{(n-1)}} \delta_{\lambda_r^{(m-1)} \lambda_s^{(m-1)}}.$$

We can repeat this with the complete set of Casimirs of $S_{n-1} \times S_{m-1}$, allowing us to conclude that

$$\langle (r_n, r_m)'_I, i | \Gamma_R\big((ai)\big) | (s_n, s_m)'_I \rangle \propto \delta_{(r_n, r_m)'_I (s_n, s_m)'_I}.$$

$\delta_{(r_n, r_m)'_I (s_n, s_m)'_I}$ is equal to 1 if $(r_n, r_m)'_I$ and $(s_n, s_m)'_I$ have the same shape. Of course, this is just a consequence of Schur's Lemma: since $\Gamma_R\big((ai)\big)$ commutes with all of the elements of the $S_{n-1} \times S_{m-1}$ subgroup, it is proportional to the identity when acting on any irreducible representation of $S_{n-1} \times S_{m-1}$. Thus, (the label i inside the ket labels the carrier space state)

$$\langle (r_n, r_m)'_I, j | \Gamma_R\big((ai)\big) | (r_n, r_m)'_I, i \rangle = \eta_{(r_n, r_m)'_I}\delta_{ij}.$$

_____

[2]It is only when (r_n, r_m) and (s_n, s_m) have the same shape that the trace $\text{Tr}_{(r_n, r_m)(s_n, s_m)}$ has any meaning.

If we allow $\Gamma_R\big((ai)\big)$ to act in the full carrier space of R, this equation is modified to (the sum is over all irreducible representations (t_n, t_m) that can subduce a $(t_n, t_m)'_K$ of the same shape as $(r_n, r_m)'_I$; the $\delta_{(r_n, r_m)'_I (s_n, s_m)'_J}$ on the last line is there to remind the reader that a non-zero result is obtained only if $(r_n, r_m)'_I$ and $(s_n, s_m)'_J$ have the same shape)

$$\langle (s_n, s_m)'_J, j | \Gamma_R\big((ai)\big) | (r_n, r_m)'_I, i \rangle = \sum_{(t_n, t_m)'_K} \eta^{(t_n, t_m)'_K}_{(r_n, r_m)'_I} \langle (s_n, s_m)'_J, j | (t_n, t_m)'_K, i \rangle$$
$$= \eta^{(s_n, s_m)'_J}_{(r_n, r_m)'_I} \delta_{(r_n, r_m)'_I (s_n, s_m)'_J} \delta_{ij}$$

Although $(s_n, s_m)'_J$ and $(r_n, r_m)'_I$ have the same shape, they may be distinct representations in which case they were subduced by *different* $S_n \times S_m$ representations. These matrix elements are needed to provide a complete generalization of strand diagrams. This has an important implication: in the present case, the most general non-vanishing strand diagrams will not only record the reordering of boxes in the row and column states, it will also allow the shape of the row state $S_n \times S_m$ representation to reorder itself (by the movement of a single box in r_n and/or a single box in r_m) into the column state $S_n \times S_m$ representation.

The trace we are interested in can also be expressed in terms of $\eta^{(r_n, r_m)'_I}_{(r_n, r_m)'_J}$. Indeed, it is clear that

$$\mathrm{Tr}_{(r_n, r_m)(s_n, s_m)}\Big(\Gamma_R\big((ai)\big)\Big) = \sum_I \mathrm{Tr}_{(r_n, r_m)'_I (s_n, s_m)'_I}\Big(\Gamma_R\big((ai)\big)\Big)$$
$$= \sum_I \eta^{(r_n, r_m)'_I}_{(r_n, r_m)'_I} d_{(r_n, r_m)'_I}.$$

$d_{(r_n, r_m)'_I}$ is the dimension of the irreducible representation $(r_n, r_m)'_I$. Since (r_n, r_m) and (s_n, s_m) have the same shape, we will not need to worry about the extra complication of changing the $S_n \times S_m$ representation between the row and column states.

To determine the restricted character we need, we now only need to fix the $\eta^{(r_n, r_m)'_I}_{(r_n, r_m)'_I}$. The most direct way we have found to do this proceeds by determining the explicit change of basis from a natural basis of S_{n+m} to a natural basis for the $S_n \times S_m$ subgroup. We will illustrate the method with an example: using the methods of the last subsection, we easily find

$$\mathrm{Tr}_{\square\square,\square\square}\left(\Gamma_{\square\square}\big((3, 4)\big)\right) = -\frac{1}{3}.$$

To extract $\square\square$ from $\square\square$ we need to pull off the last box in the first row and the last box in the first column. They can be pulled off in any order, so that we can write (i labels the states in the $\square\square$ carrier space)

$$| \square\square ; \square\square i, \square\square \rangle = \alpha | \square\square i \rangle + \beta | \square\square i \rangle. \tag{3.2}$$

Because the above state is normalized, we need $\alpha^2 + \beta^2 = 1$. Next, using appendix D.2

– 11 –

of [14] we can write

$$\Gamma_{\tiny\square}((45))\,|{\tiny\square};{\tiny\square}\,i,{\tiny\square}\rangle = |{\tiny\square};{\tiny\square}\,i,{\tiny\square}\rangle,$$

$$\Gamma_{\tiny\square}((45))\left[\alpha|{\tiny\square}\,i\rangle + \beta|{\tiny\square}\,i\rangle\right] = \left(\frac{\alpha}{4}+\beta\frac{\sqrt{15}}{4}\right)|{\tiny\square}\,i\rangle + \left(-\frac{\beta}{4}+\alpha\frac{\sqrt{15}}{4}\right)|{\tiny\square}\,i\rangle.$$

This then implies two equations

$$\frac{\alpha}{4}+\beta\frac{\sqrt{15}}{4}=\alpha, \qquad -\frac{\beta}{4}+\alpha\frac{\sqrt{15}}{4}=\beta.$$

They are not independent (as expected) and imply $\alpha=\sqrt{\frac{5}{3}}\beta$ so that

$$\alpha=\sqrt{\frac{5}{8}}, \qquad \beta=\sqrt{\frac{3}{8}}.$$

It is now straight forward to use the standard strand diagram techniques of [16] to verify that

$$\sum_i\left(\sqrt{\frac{5}{8}}\langle{\tiny\square}\,i| + \sqrt{\frac{3}{8}}\langle{\tiny\square}\,i|\right)\Gamma_{\tiny\square}\big((3,4)\big)\left(\sqrt{\frac{5}{8}}|{\tiny\square}\,i\rangle + \sqrt{\frac{3}{8}}|{\tiny\square}\,i\rangle\right) = -\frac{1}{3}.$$

Although we have computed an on the diagonal character, it is clear that once the relationship (3.2) is established, it can be used to compute off the diagonal block characters by employing standard strand diagrams.

Summary of the logic. Using the action of 2-cycles from the S_m subgroup - something we already know in both bases - we have been able to determine the explicit change of basis from a natural basis of S_{n+m} to a natural basis for the $S_n \times S_m$ subgroup. This has enabled us to compute the restricted character of the two cycle $(n,n+1)$ which "straddles" S_n and S_m. Thus, we can now compute the restricted characters of all two cycles of the form $(i,i+1)$, which is all that is needed to compute the restricted character of a general group element.

We will now argue that, as long as the number of boxes in r_m is small, we can find formulas for the above change of basis for any representation r_n. Removing the m boxes (used to build r_m) from the S_{n+m} representation R gives a set of states that carry an index for the carrier space of r_n. The coefficient describing the change of basis is clearly independent of this index. Thus, in what follows, the carrier space index of representation r_n plays no role and is hence suppressed.

Restricted character of $(n,n+1)$ for $m=1$. Denote the two labels of the restricted trace by $R,(r_n,\square)$ and $R,(s_n,\square)$. R is a Young diagram with $n+1$ boxes; r_n and s_n are both Young diagrams with n boxes; they are both obtained by removing a single box from R. Denote the weight of the box that must be removed from R to obtain r_n by c_r; denote the weight of the box that must be removed from R to obtain s_n by c_s. For an off the

diagonal block, $c_r \neq c_s$. Let R'' denote the Young diagram obtained when both boxes are removed. It is now straight forward to show that

$$\mathrm{Tr}_{\,(r_n,\Box)(s_n,\Box)} \left(\Gamma_R\big((n,n+1)\big) \right) = \sqrt{1 - \frac{1}{(c_r - c_s)^2}}\, d_{R''},$$

where $d_{R''}$ is the dimension of R''.

Restricted character of $(n, n+1)$ for $m = 2$. The two possible types of restricted Schur labels are $R, (r_n, \Box\Box)$ and $R, (r_n, \Box)$. R is a Young diagram with $n+2$ boxes; r_n is a Young diagram with n boxes. r_n is obtained from R by removing two boxes from R. We say that box a is above box b if the weight of box a is greater than the weight of box b. Denote the weights of the two boxes as c_1 and c_2, such that the box with weight c_1 is above the box with weight c_2. The relations between the natural S_{n+2} basis and the natural $S_n \times S_2$ bases are (although we have written these relations using a particular Young diagram, they are true in general; on the right hand side, the box with label a is to be removed first)

$$\left| \begin{smallmatrix}\Box\Box\\\Box\end{smallmatrix}; \Box\, i, \Box\Box \right\rangle = \sqrt{\frac{c_1 - c_2 + 1}{2(c_1 - c_2)}} \left| \begin{smallmatrix}\Box\Box\,^a\\\Box\,^b\end{smallmatrix}\, i \right\rangle + \sqrt{\frac{c_1 - c_2 - 1}{2(c_1 - c_2)}} \left| \begin{smallmatrix}\Box\Box\,^b\\\Box\,^a\end{smallmatrix}\, i \right\rangle,$$

$$\left| \begin{smallmatrix}\Box\Box\\\Box\end{smallmatrix}; \Box\, i, \Box \right\rangle = \sqrt{\frac{c_1 - c_2 - 1}{2(c_1 - c_2)}} \left| \begin{smallmatrix}\Box\Box\,^a\\\Box\,^b\end{smallmatrix}\, i \right\rangle - \sqrt{\frac{c_1 - c_2 + 1}{2(c_1 - c_2)}} \left| \begin{smallmatrix}\Box\Box\,^b\\\Box\,^a\end{smallmatrix}\, i \right\rangle.$$

Notice that these states are orthogonal as they must be. It is now straight forward to obtain any particular character we want by employing standard strand diagram methods.

We will now consider a particular example. The labels for the restricted trace are $R, (r_n, \Box\Box)$ and $R, (s_n, \Box\Box)$. To obtain a non-zero off the diagonal restricted character, one of the boxes removed from R to obtain r_n must be in the same position as one of the boxes removed from R to obtain s_n. Assume that the common box has weight c_1. Denote the weight of the second box that must be removed to obtain r_n by c_2 and denote the weight of the second box that must be removed to obtain s_n by c_2^*. Denote the Young diagram obtained by removing all three boxes from R by R'''. It is straight forward to show that

$$\mathrm{Tr}_{\,(r_n,\Box\Box)(s_n,\Box\Box)} \left(\Gamma_R\big((n,n+1)\big) \right) = \sqrt{\frac{c_1 - c_2 + 1}{2(c_1 - c_2)}} \sqrt{\frac{c_1 - c_2^* + 1}{2(c_1 - c_2^*)}} \sqrt{1 - \frac{1}{(c_2 - c_2^*)^2}}\, d_{R'''}.$$

Restricted character of $(n, n+1)$ for $m = 3$. We will discuss this example in some detail because it will provide the key to obtaining general results. We will consider the case in which the three boxes to be removed have no sides in common. In this case, the three possible types of restricted Schur labels are $R, (r_n, \Box\Box\Box)$, $R, (r_n, \begin{smallmatrix}\Box\Box\\\Box\end{smallmatrix})$ and $R, (r_n, \begin{smallmatrix}\Box\\\Box\\\Box\end{smallmatrix})$. R is a Young diagram with $n+3$ boxes; r_n is a Young diagram with n boxes. r_n is obtained from R by removing three boxes from R. Denote the weights of the boxes to be removed by c_1, c_2 and c_3. The box with weight c_1 lies above the boxes with weights c_2 and c_3; the box with weight c_2 lies above the box with weight c_3. Consider the expansion (on the right

hand side, the box with label a is to be removed first the box with label b second and the box with label c third)

$$|\,{}^{\boxplus}\!\!;{}^{\boxplus}\!\!\boxslash\rangle = \alpha_{123}|{}^{\boxplus}_{c}\rangle + \alpha_{132}|{}^{\boxplus}_{b}\rangle + \alpha_{213}|{}^{\boxplus}_{c}\rangle$$
$$+\alpha_{231}|{}^{\boxplus}_{a}\rangle + \alpha_{312}|{}^{\boxplus}_{b}\rangle + \alpha_{321}|{}^{\boxplus}_{a}\rangle\,.$$

Notice that the subscripts encode the order in which boxes are to be dropped. By considering the action of the cycle of $(n+2, n+3)$ on the above expression, we obtain 6 equations. Only three of these are independent; they read

$$\alpha_{123} = \frac{1}{c_1 - c_2}\alpha_{123} + \sqrt{1 - \frac{1}{(c_1 - c_2)^2}}\alpha_{213},$$

$$\alpha_{132} = \frac{1}{c_1 - c_3}\alpha_{132} + \sqrt{1 - \frac{1}{(c_1 - c_3)^2}}\alpha_{231},$$

$$\alpha_{321} = \frac{1}{c_3 - c_2}\alpha_{321} + \sqrt{1 - \frac{1}{(c_3 - c_2)^2}}\alpha_{312}.$$

Similarly, by considering the action of the cycle $(n+1, n+2)$ we obtain the following three independent equations

$$\alpha_{123} = \frac{1}{c_2 - c_3}\alpha_{123} + \sqrt{1 - \frac{1}{(c_2 - c_3)^2}}\alpha_{132},$$

$$\alpha_{213} = \frac{1}{c_1 - c_3}\alpha_{213} + \sqrt{1 - \frac{1}{(c_1 - c_3)^2}}\alpha_{312},$$

$$\alpha_{321} = \frac{1}{c_2 - c_1}\alpha_{321} + \sqrt{1 - \frac{1}{(c_2 - c_1)^2}}\alpha_{231}.$$

These six equations can be written in a very compact form: let p denote the subscript of a particular coefficient, i.e. a particular ordering of the three numbers 1, 2 and 3. Define the action of the cycle $(i, i+1)$ on p as follows: all numbers not equal to i or $i+1$ stay where they are; the numbers i and $i+1$ swap positions. Thus, $(1,2) \cdot 123 = 213$. We will also index the entries of p by $i = 1, 2, 3$. Thus, for $p = 231$, we have $p(1) = 2$, $p(2) = 3$ and $p(3) = 1$. The six equations above can now be written as

$$\alpha_p = \frac{1}{c_{p^{-1}(i)} - c_{p^{-1}(i+1)}}\alpha_p + \sqrt{1 - \frac{1}{(c_{p^{-1}(i)} - c_{p^{-1}(i+1)})^2}}\alpha_{(i,i+1)\cdot p}, \qquad i = 1, 2, \ldots, m-1. \quad (3.3)$$

Simple algebra gives

$$\alpha_p = \sqrt{\frac{c_{p^{-1}(i)} - c_{p^{-1}(i+1)} + 1}{c_{p^{-1}(i)} - c_{p^{-1}(i+1)} - 1}}\alpha_{(i,i+1)\cdot p}\,.$$

These equations completely determine the unknown coefficients. Indeed, each subscript is a particular ordering of the three numbers 1, 2 and 3. By using the first set of equations

we can swap the positions of 1 and 2; by using the second set we can swap the positions of 2 and 3. Using these two operations we can relate any coefficient to any other. Thus, the normalization condition

$$\alpha_{123}^2 + \alpha_{132}^2 + \alpha_{213}^2 + \alpha_{231}^2 + \alpha_{312}^2 + \alpha_{321}^2 = 1$$

can be written completely in terms of α_{123} (say). This determines α_{123} and hence all of the coefficients. Solving for α_{123} we obtain

$$\alpha_{123} = \sqrt{\frac{(c_1 - c_2 + 1)(c_1 - c_3 + 1)(c_2 - c_3 + 1)}{6(c_1 - c_2)(c_1 - c_3)(c_2 - c_3)}}.$$

These conclusions are rather general: the equations following from applications of two cycles of the form $(n+i, n+i+1)$, $i = 1, 2, \ldots, m-1$ are given by (3.3) for any state on the right hand side with label $R, (r_n, (m))$ where (m) is the completely symmetric representation. For any state on the right hand side, the equations following from application of the two cycles $(i, i+1)$, $i = 1, 2, \ldots, m-1$ together with the normalization condition determines the expansion coefficients uniquely. The resulting solution is

$$\alpha_{123\cdots m} = \sqrt{\frac{1}{m!} \prod_{i=1}^{m-1} \left[\prod_{j=i+1}^{m} \frac{c_i - c_j + 1}{c_i - c_j} \right]}.$$

It is now straight forward to compute restricted characters for any restricted traces with labels $R, (r_n, \square\square\square)$.

Next, consider restricted traces with label $R, (r_n, \begin{smallmatrix}\square\\\square\end{smallmatrix})$. Things work exactly as for the case just considered. The relevant expansion is

$$\left| \begin{smallmatrix}\square\square\\\square\end{smallmatrix} ; \begin{smallmatrix}\square\square\\\square\end{smallmatrix} \right\rangle = \beta_{123} \left| \begin{smallmatrix}\square\square{}^a\\\square{}^b\\\square{}^c\end{smallmatrix} \right\rangle + \beta_{132} \left| \begin{smallmatrix}\square\square{}^a\\\square{}^c\\\square{}^b\end{smallmatrix} \right\rangle + \beta_{213} \left| \begin{smallmatrix}\square\square{}^b\\\square{}^a\\\square{}^c\end{smallmatrix} \right\rangle$$

$$+ \beta_{231} \left| \begin{smallmatrix}\square\square{}^b\\\square{}^c\\\square{}^a\end{smallmatrix} \right\rangle + \beta_{312} \left| \begin{smallmatrix}\square\square{}^c\\\square{}^a\\\square{}^b\end{smallmatrix} \right\rangle + \beta_{321} \left| \begin{smallmatrix}\square\square{}^c\\\square{}^b\\\square{}^a\end{smallmatrix} \right\rangle.$$

We will write the results of our analysis for general m. It is straight forward to obtain

$$-\beta_p = \frac{1}{c_{p^{-1}(i)} - c_{p^{-1}(i+1)}} \beta_p + \sqrt{1 - \frac{1}{(c_{p^{-1}(i)} - c_{p^{-1}(i+1)})^2}} \beta_{(i,i+1)\cdot p}, \qquad i = 1, 2, \ldots, m-1,$$

(3.4)

and hence

$$\beta_p = -\sqrt{\frac{c_{p^{-1}(i)} - c_{p^{-1}(i+1)} - 1}{c_{p^{-1}(i)} - c_{p^{-1}(i+1)} + 1}} \beta_{(i,i+1)\cdot p}.$$

Solving these equations together with the normalization condition gives

$$\beta_{123\cdots m} \sqrt{\frac{1}{m!} \prod_{i=1}^{m-1} \left[\prod_{j=i+1}^{m} \frac{c_i - c_j - 1}{c_i - c_j} \right]}.$$

As a check of our phase conventions, we have checked that α_p and β_p define orthogonal vectors.

We have not been able to treat the last case, restricted traces with labels $R,(r_n,\boxebox)$, for general r_m. Further, the solution we have obtained is not unique, because the representation appears twice in the outer product $\Box \times \Box \times \Box$. We have tried to use the freedom we have to choose the simplest possible solution. Consider the expansion

$$|\,\Box\,;\Box\Box\,i\rangle^a = \gamma^a_{123\,i}|\begin{smallmatrix}\Box^a\\b\\c\end{smallmatrix}\rangle + \gamma^a_{132\,i}|\begin{smallmatrix}\Box^a\\c\\b\end{smallmatrix}\rangle + \gamma^a_{213\,i}|\begin{smallmatrix}\Box^b\\c\\a\end{smallmatrix}\rangle$$
$$+\gamma^a_{231\,i}|\begin{smallmatrix}\Box^b\\a\\c\end{smallmatrix}\rangle + \gamma^a_{312\,i}|\begin{smallmatrix}\Box^c\\b\\a\end{smallmatrix}\rangle + \gamma^a_{321\,i}|\begin{smallmatrix}\Box^c\\a\\b\end{smallmatrix}\rangle.$$

a is a multiplicity label - it takes the values $1,2$. i is a label for states in the carrier space of r_m; it takes one of the two values

$$i = 1 \leftrightarrow \begin{smallmatrix}3&2\\1\end{smallmatrix}\,, \qquad i = 2 \leftrightarrow \begin{smallmatrix}3&1\\2\end{smallmatrix}\,.$$

A straight forward (but tedious) computation now determines

$$\gamma^1_{123\,1} = -\frac{(c_{12}-2)\sqrt{(c_{23}+1)(c_{23}-1)(c_{13}+1)}}{d}, \qquad \gamma^1_{132\,1} = \sqrt{\frac{c_{23}-1}{c_{23}+1}}\gamma^1_{123\,1},$$

$$\gamma^1_{213\,1} = 2\frac{\sqrt{(c_{12}-1)(c_{12}+1)(c_{23}+1)(c_{23}-1)(c_{13}+1)}}{d}, \qquad \gamma^1_{312\,1} = \sqrt{\frac{c_{13}-1}{c_{13}+1}}\gamma^1_{213\,1},$$

$$\gamma^1_{231\,1} = -\frac{(c_{12}+1)(c_{23}+2)\sqrt{(c_{13}-1)}}{d}, \qquad \gamma^1_{321\,1} = \sqrt{\frac{c_{12}-1}{c_{12}+1}}\gamma^1_{231\,1},$$

$$\gamma^1_{123\,2} = \frac{c_{12}\sqrt{3(c_{23}+1)(c_{23}-1)(c_{13}+1)}}{d}, \qquad \gamma^1_{132\,2} = -\sqrt{\frac{c_{23}+1}{c_{23}-1}}\gamma^1_{123\,2},$$

$$\gamma^1_{213\,2} = 0, \qquad \gamma^1_{312\,2} = 0,$$

$$\gamma^1_{231\,2} = -\frac{(c_{12}-1)c_{23}\sqrt{3(c_{13}-1)}}{d}, \qquad \gamma^1_{321\,2} = -\sqrt{\frac{c_{12}+1}{c_{12}-1}}\gamma^1_{231\,2},$$

$$\gamma^2_{123\,1} = \frac{(c_{23}+1)\sqrt{3(c_{12}+1)(c_{12}-1)(c_{13}-1)}}{d}, \qquad \gamma^2_{132\,1} = \sqrt{\frac{c_{23}-1}{c_{23}+1}}\gamma^2_{123\,1},$$

$$\gamma^2_{213\,1} = -\frac{(c_{13}+1)\sqrt{3(c_{13}-1)}}{d}, \qquad \gamma^2_{312\,1} = \sqrt{\frac{c_{13}-1}{c_{13}+1}}\gamma^2_{213\,1},$$

$$\gamma^2_{231\,1} = -\frac{\sqrt{3(c_{12}+1)(c_{12}-1)(c_{23}+1)(c_{23}-1)(c_{13}+1)}}{d}, \qquad \gamma^2_{321\,1} = \sqrt{\frac{c_{12}-1}{c_{12}+1}}\gamma^2_{231\,1},$$

$$\gamma^2_{123\,2} = \frac{(c_{23}-1)\sqrt{(c_{12}+1)(c_{12}-1)(c_{13}-1)}}{d}, \qquad \gamma^2_{132\,2} = -\sqrt{\frac{c_{23}+1}{c_{23}-1}}\gamma^2_{123\,2},$$

$$\gamma^2_{213\,2} = \frac{(2c_{12}c_{23}+c_{12}-c_{23}+1)\sqrt{c_{13}-1}}{d}, \qquad \gamma^2_{312\,2} = -\sqrt{\frac{c_{13}+1}{c_{13}-1}}\gamma^2_{213\,2},$$

$$\gamma^2_{231\,2} = \frac{\sqrt{(c_{12}+1)(c_{12}-1)(c_{23}+1)(c_{23}-1)(c_{13}+1)}}{d}, \qquad \gamma^2_{321\,2} = -\sqrt{\frac{c_{12}+1}{c_{12}-1}}\gamma^2_{231\,2},$$

where $c_{ij} \equiv c_i - c_j$ and

$$d = \sqrt{6(c_1-c_2)(c_2-c_3)(c_1-c_3)(2(c_1-c_2)(c_2-c_3)-2c_2+c_3+c_1+1)}.$$

As a partial check of our phase conventions we have verified that

$$\{|\;\substack{\square\square\\\square}\;;\substack{\square\square\\\square}\;\square\square\square\rangle\,,\;|\;\substack{\square\square\\\square}\;;\substack{\square\square\\\square}\;\square\square\rangle\,,\;|\;\substack{\square\square\\\square}\;;\square\square\;\square\square_i\rangle^a\}$$

provides an orthonormal basis. This does not yet provide a complete treatment of the $m = 3$ case, because we have not yet considered the case that the three boxes removed share common sides. The generalization to this case is straight forward, using the methods we developed in this section.

3.3 Some example computations of restricted Littlewood-Richardson coefficients

In this subsection we will discuss the computation of the restricted Littlewood-Richardson coefficients that will be needed to study graviton interactions in the next section. The first case we wish to consider are representations where R, r_n and r_m are totally antisymmetric. In the case of a single matrix, these correlators are dual to branes that have expanded in the S^5 of the $AdS_5 \times S^5$ geometry. Since $n + m$ is cut off at N it is natural to conjecture that the same interpretation holds even when the operator is built using two complex Higgs fields. As an example, for $n = 2$ and $m = 3$ we are discussing the restricted representation with labels

$$R = \substack{\square\\\square\\\square\\\square\\\square} \equiv (1^5), \qquad (r_n, r_m) = (\substack{\square\\\square},\substack{\square\\\square}) \equiv (1^3, 1^2).$$

This particular class of restricted characters are particularly easy to compute: because the dimension of the representation and the dimensions of the restriction are equal, the restriction is trivial and we have ($\epsilon(\sigma)$ is 1 if σ is an odd permutation and 0 if σ is an even permutation)

$$\chi_{(1^{n+m}),(1^n,1^m)}(\sigma) = \chi_{(1^{n+m})}(\sigma) = (-1)^{\epsilon(\sigma)},$$

and

$$\chi^{(1^{n+m}),(1^n,1^m)}(\sigma) = \frac{n!m!}{(n+m)!}\chi_{1^{n+m}}(\sigma).$$

Consequently, for the case under consideration, we can express the restricted Littlewood-Richardson numbers in terms of the Littlewood-Richardson numbers as

$$f^{1^{n_1+n_2+m_1+m_2},(1^{n_1+n_2},1^{m_1+m_2})}_{1^{n_1+m_1},(1^{n_1},1^{m_1})\,1^{n_2+m_2},(1^{n_2},1^{m_2})} = \frac{n_{12}!m_{12}!(n_1+m_1)!(n_2+m_2)!}{n_1!n_2!m_1!m_2!(n_{12}+m_{12})!}f_{1^{n_1+m_1}\,1^{n_2+m_2}\,1^{n_1+n_2+m_1+m_2}},$$

where we have used the formula

$$f_{RST} = \frac{1}{n_R!n_S!}\sum_{\alpha_1\in S_R}\sum_{\alpha_2\in S_S}\chi_R(\sigma_1)\chi_S(\sigma_2)\chi_T(\sigma_1\circ\sigma_2).$$

It is satisfying that upon setting $m_1 = m_2 = 0$ our restricted Littlewood-Richardson numbers reproduce the standard Littlewood-Richardson numbers. Rules for the computation of the Littlewood-Richardson coefficients can be found, for example, in [28]. For the case under consideration here, the Littlewood-Richardson number is 1, so that

$$f^{1^{n_1+n_2+m_1+m_2},(1^{n_1+n_2},1^{m_1+m_2})}_{1^{n_1+m_1},(1^{n_1},1^{m_1})\,1^{n_2+m_2},(1^{n_2},1^{m_2})} = \frac{n_{12}!m_{12}!(n_1+m_1)!(n_2+m_2)!}{n_1!n_2!m_1!m_2!(n_{12}+m_{12})!}.$$

Notice that these restricted Littlewood-Richardson numbers are all ≤ 1.

A second case of interest, are representations where R, r_n and r_m are totally symmetric. The corresponding single matrix correlators were conjectured to be dual to branes that have expanded in the AdS_5 of the $AdS_5 \times S^5$ geometry. A beautiful test of this conjecture was given in [29]. We again conjecture that the same interpretation holds even when the operator is built using two complex Higgs fields. Again, this class of restricted characters are easy to compute because the dimension of the representation and the dimension of the restriction are equal. Exactly as above, we can express the restricted Littlewood-Richardson numbers in terms of the Littlewood-Richardson numbers as

$$
\begin{aligned}
f^{(n_{12}+m_{12}),((n_1+n_2),(m_1+m_2))}_{(n_1+m_1),((n_1),(m_1))\,(n_2+m_2),((n_2),(m_2))} &= \frac{n_{12}!m_{12}!(n_1+m_1)!(n_2+m_2)!}{n_1!n_2!m_1!m_2!(n_{12}+m_{12})!} f_{(n_1+m_1)\,(n_2+m_2)\,(n_{12}+m_{12})} \\
&= \frac{n_{12}!m_{12}!(n_1+m_1)!(n_2+m_2)!}{n_1!n_2!m_1!m_2!(n_{12}+m_{12})!},
\end{aligned}
$$

where we have used the fact that in this case the Littlewood-Richardson number is again 1. Again, upon setting $m_1 = m_2 = 0$ our restricted Littlewood-Richardson numbers reproduce the standard Littlewood-Richardson numbers.

Another case in which general results can be obtained, involves multiplying a polynomial in Z ($\chi_{R_1}(Z)$ with R_1 an irreducible representation of S_n) by a polynomial in X ($\chi_{R_2}(X)$ with R_2 an irreducible representation of S_m). The relevant restricted Littlewood-Richardson numbers are simply

$$
\begin{aligned}
f^{R,(r_n,r_m)}_{R_1,(R_1,\cdot)\,R_2,(\cdot,R_2)} &= \frac{d_R}{(n+m)!d_{r_n}d_{r_m}} \sum_{\sigma_1 \in S_n} \sum_{\sigma_2 \in S_m} \chi_{R_1}(\sigma_1)\chi_{R_2}(\sigma_2)\chi_{R,(r_n,r_m)}(\sigma_1 \circ \sigma_2) \\
&= \frac{d_R}{(n+m)!d_{r_n}d_{r_m}} \sum_{\sigma_1 \in S_n} \chi_{R_1}(\sigma_1)\chi_{r_n}(\sigma_1) \sum_{\sigma_2 \in S_m} \chi_{R_2}(\sigma_2)\chi_{r_m}(\sigma_2) \\
&= \frac{n!m!}{(n+m)!} \frac{d_R}{d_{r_n}d_{r_m}} \delta_{R_1 r_n} \delta_{R_2 r_m}.
\end{aligned}
$$

We have not explicitly indicated a multiplicity label for r_n and r_m. This label is needed if there is more than one way to subduce (r_n, r_m) from R.

Finally, if we consider products of polynomials for which the total number of Xs participating is small, we can construct the explicit change of basis developed in the previous subsection. With this change of basis in hand, all restricted characters appearing in the formula for the restricted Littlewood-Richardson numbers can be computed using standard strand diagram techniques.

4. Graviton interactions

A description of perturbative quantum gravity phrased in terms of graviton degrees of freedom is expected to fail at high enough energy, when the gravitons become strongly interacting. A perturbative description of the high energy physics needs to be phrased in terms of new degrees of freedom that are weakly interacting at high energy. The exact computations of [22], performed in the half-BPS sector of IIB string theory on $AdS_5 \times S^5$

show that the sphere giant gravitons provide satisfactory degrees of freedom for a high energy description. The half-BPS dynamics is captured by the dynamics of a single matrix. In this section, using the methods we have developed above, we will extend the results of [22] beyond the one matrix sector. Our conclusions are consistent with those of [22].

To estimate where the graviton description breaks down, we will estimate where the two point functions differ appreciably from their planar limit. This difference indicates that non-planar corrections are no longer suppressed and this implies that the large N orthogonality of the trace basis fails. The trace basis is dual, at low energies, to the supergravity Fock space. The fact that orthogonality fails tells us that there are non-negligible interactions mixing the graviton Fock space states; these interactions will invalidate perturbation theory. If we build traces that contain only Xs or only Zs, we know that orthogonality fails when we have $O(\sqrt{N})$ fields in each trace [25, 26]. Is this conclusion still valid when we consider traces containing both Xs and Zs? The computations of [25, 26] used the fact that the combinatorics associated with the Wick contractions can be computed using a zero dimensional complex matrix model. These techniques do not have an easy extension to the mixed traces that interest us. A more powerful alternative technique (employing symmetric group theory methods) to compute the single trace correlators was developed in [27]. Using the logic of [27], we are able to compute mixed trace correlators. The basic idea is that using the results of [11], we know how to compute mixed correlators in the restricted Schur basis. To get correlators of traces we simply need to change from the restricted Schur basis to the trace basis. This change of basis is easily accomplished using dual characters. Any single trace operator can be written as

$$\mathrm{Tr}\left(\sigma_c Z^{\otimes n} \otimes X^{\otimes m}\right) = \sum_{R,(r_n,r_m)} \chi^{R,(r_n,r_m)}(\sigma_c)\chi_{R,(r_n,r_m)}(Z,X),$$

where σ_c is an $n+m$ cycle. For concreteness we will consider the trace $\mathrm{Tr}\left(Z^n X^m\right)$ which corresponds to $\sigma_c = (1\,2\,3\,\ldots\,m+n-2\,m+n-1\,m+n)$, up to conjugation by an element of $S_n \times S_m$. The computation of the relevant restricted characters is carried out in appendix D. We find that this character vanishes unless r_n and r_m are both hooks - Young diagrams with at most a single column whose length is > 1. Denote the number of rows in r_n by $s_n + 1$ and the number of rows in r_m by $s_m + 1$. To subduce (r_n, r_m), the Young diagram R must have at most two column with lengths > 2. We again imagine that boxes are removed from R to compose r_m. The boxes that remain form r_n. The result of appendix D says

$$\chi_{R,(r_n,r_m)}(\sigma_c) = \frac{(-1)^{s_n+s_m}}{mn}\left(\sum_i c_i - mN - \frac{m(m-2s_m-1)}{2}\right),$$

where the sum on i runs over all boxes removed from R to produce r_m. Thus, the dual character is

$$\chi^{R,(r_n,r_m)}(\sigma_c) = \frac{(\mathrm{hooks})_{r_n}(\mathrm{hooks})_{r_m}}{(\mathrm{hooks})_R}\frac{(-1)^{s_n+s_m}}{mn}\left(\sum_i c_i - mN - \frac{m(m-2s_m-1)}{2}\right)$$
$$= s_n!(n-s_n-1)!s_m!(m-s_m-1)!(-1)^{s_n+s_m} \times \mathcal{F},$$

where the R dependent factor $\mathcal{F}$ is given by

$$\mathcal{F} = \frac{\left(\sum_i c_i - mN - \frac{m(m-2s_m-1)}{2}\right)}{(\text{hooks})_R}.$$

Using the two point functions of [11] it is now straight forward to obtain

$$\langle \text{Tr}\left(\sigma_c Z^{\otimes n} \otimes X^{\otimes m}\right)^\dagger \text{Tr}\left(\sigma_c Z^{\otimes n} \otimes X^{\otimes m}\right)\rangle = \sum_{R,(r_n,r_m)} \frac{s_m!(m-s_m-1)!s_n!(n-s_n-1)!}{mn}$$

$$\times \text{Dim}_R \left(\sum_i c_i - mN - \frac{m(m-2s_m-1)}{2}\right)^2,$$

where Dim_R is the dimension of $\text{SU}(N)$ representation R and the sum runs over all labels such that r_n and r_m are hooks. Denote the number of boxes in the first row of R by r_1 and the number of boxes in the second row of R by r_2. Denote the number of boxes in the first column of R by c_1 and the number of boxes in the second column of R by c_2. It is straight forward to see that

$$r_2 = m+n-s_m-s_n-r_1+1, \qquad c_2 = s_n+s_m-3-c_1, \qquad r_1+r_2+c_1+c_2 = n+m+4.$$

Summing over $R,(r_n,r_m)$ can be replaced by a sum over s_n, s_m, r_1 and c_1. The value of the sum is

$$\langle \text{Tr}\left(\sigma_c Z^{\otimes n} \otimes X^{\otimes m}\right)^\dagger \text{Tr}\left(\sigma_c Z^{\otimes n} \otimes X^{\otimes m}\right)\rangle = \frac{N^2}{(n+1)(m+1)(N^2-1)}$$

$$\times \left(\frac{1}{(n+2)(m+2)}\left[\frac{(N+n+1)!}{N!} - \frac{(N-1)!}{(N-n-2)!}\right]\left[\frac{(N+m+1)!}{N!} - \frac{(N-1)!}{(N-m-2)!}\right]\right.$$

$$- \frac{1}{N(m+2)}\left[\frac{(N+n)!}{N!} - \frac{(N-1)!}{(N-n-1)!}\right]\left[\frac{(N+m+1)!}{N!} - \frac{(N-1)!}{(N-m-2)!}\right]$$

$$- \frac{1}{N(n+2)}\left[\frac{(N+n+1)!}{N!} - \frac{(N-1)!}{(N-n-2)!}\right]\left[\frac{(N+m)!}{N!} - \frac{(N-1)!}{(N-m-1)!}\right]$$

$$\left.+ \left[\frac{(N+n)!}{N!} - \frac{(N-1)!}{(N-n-1)!}\right]\left[\frac{(N+m)!}{N!} - \frac{(N-1)!}{(N-m-1)!}\right]\right).$$

The above result is exact (i.e. to all orders in N) in the free field theory limit. Following [22] we make us of the identity

$$\frac{(N+p_1)!}{(N-p_0-1)!} = N^{p_0+p_1+1} \exp\left[\frac{1}{2N}(p_1-p_0)(p_0+p_1+1) + O(N^{-2})\right],$$

valid when $p_0 \ll N$ and $p_1 \ll N$, to explore the behavior of our two point function. We find an appreciable difference from the planar answer (which is N^{n+m}) when n^2/N or m^2/N (or both) are held fixed. The leading behaviour of our two point correlation function is

$$\langle \text{Tr}\left(\sigma_c Z^{\otimes n} \otimes X^{\otimes m}\right)^\dagger \text{Tr}\left(\sigma_c Z^{\otimes n} \otimes X^{\otimes m}\right)\rangle = \frac{4N^{n+m+2}}{nm} \sinh\frac{n^2}{2N} \sinh\frac{m^2}{2N}.$$

We have dropped terms of order $\frac{m}{N}$ and $\frac{n}{N}$. No finite order calculation in $1/N$ will reproduce this result - in this range of values for m, n perturbation theory breaks down. This suggests that, even when more than one matrix species participates in the trace, the validity of perturbation theory in terms of gravitons (that is single trace operators) breaks down when there are $O(\sqrt{N})$ matrices in the trace. The cautious reader might object that we have not built operators that correspond to chiral primaries of the SYM and hence that our operators are not dual to gravitons. This is indeed true. However, the effects we study arise because the large number of non-planar contractions possible over power the usual N^{-2} suppression of higher genus effects. Thus, this effect is really only sensitive to the number of fields appearing in each trace and not the specific details of how these fields are distributed.

We will now compute the three point function for three restricted Schur polynomials. In what follows we will always employ the multi particle normalization

$$\Gamma(1; 2|1 + 2) = \frac{\langle \chi_{\{1\}} \chi_{\{2\}} \chi^{\dagger}_{\{1+2\}} \rangle}{||\chi_{\{1\}}|| \, ||\chi_{\{2\}}|| \, ||\chi_{\{1+2\}}||}$$

of [18]. Consider first the case that each restricted Schur has totally antisymmetric labels i.e. labels of the form $1^{n+m}, (1^n, 1^m)$. This corresponds to the interaction of three sphere giant gravitons. In this case it is straight forward, using the results of section 2 and 3.3 and of [11] to obtain

$$\Gamma(1; 2|1 + 2) = \frac{\langle \chi_{1^{n_1+m_1}, (1^{n_1}, 1^{m_1})} \chi_{1^{n_2+m_2}, (1^{n_2}, 1^{m_2})} \chi^{\dagger}_{1^{n_{12}+m_{12}}, (1^{n_{12}}, 1^{m_{12}})} \rangle}{||\chi_{1^{n_1+m_1}, (1^{n_1}, 1^{m_1})}|| \, ||\chi_{1^{n_2+m_2}, (1^{n_2}, 1^{m_2})}|| \, ||\chi_{1^{n_{12}+m_{12}}, (1^{n_{12}}, 1^{m_{12}})}||}$$

$$= \sqrt{\frac{(N - n_1 - m_1)!(N - n_2 - m_2)!}{N!(N - n_{12} - m_{12})!}} \sqrt{\frac{n_{12}! m_{12}!(n_1 + m_1)!(n_2 + m_2)!}{n_1! n_2! m_1! m_2!(n_{12} + m_{12})!}}.$$

We have written this result as a product of two square root factors. The first factor on the last line has the same form as the one matrix result. If $m_1 = m_2 = 0$, this factor is identically equal to the correlation function computed in the one matrix case. The growth of this term for different values of $n_1 + m_1$ and $n_2 + m_2$ has been considered in detail in [3, 22]. For both $n_1 + m_1$ and $n_2 + m_2$ fixed as $N \to \infty$ we find that [3, 22] $\Gamma(1; 2|1 + 2) \sim O(1)$, so that these restricted Schur polynomials do not provide weakly coupled degrees of freedom for long wavelength (low energy) modes. For $\frac{n_1 + m_1}{N}$ and $\frac{n_2 + m_2}{N}$ fixed and small in the limit $N \to \infty$ we find that [3, 22] $\Gamma(1; 2|1 + 2) \sim e^{-\alpha N}$ with α positive and $O(1)$: these restricted Schur polynomials provide weakly coupled degrees of freedom for high energy modes. The second factor is always ≤ 1. To see this, consider the binomial expansion of

$$(1 + x)^m = \sum_{k=0}^{m} \binom{m}{k} x^k, \qquad \text{where} \qquad \binom{m}{k} = \frac{m!}{k!(m - k)!}.$$

By comparing the coefficient of x^{r+s} coming from the expansion of $(1 + x)^m$ times the expansion of $(1 + x)^n$ to the coefficient of x^{r+s} coming from the expansion of $(1 + x)^{m+n}$, we learn that

$$\binom{m}{r}\binom{n}{s} \quad + \quad \text{non negative integers} \quad = \binom{m+n}{r+s}.$$

Thus

$$\frac{\binom{m}{r}\binom{n}{s}}{\binom{m+n}{r+s}} \leq 1,$$

which proves that the second factor is ≤ 1. Notice that when $m_1 = m_2 = 0$, the second factor is identically equal to 1 so that our result correctly reduces to the one matrix result of [3]. In summary, in the multimatrix sector we find the giant graviton degrees of freedom are weakly coupled at high energy, consistent with the conclusions of [22].

Finally, it is equally easy to compute the correlation function in the case that all three giants interacting are in completely symmetric representations. This corresponds to the interaction of three AdS giant gravitons. The result is

$$\Gamma(1;2|1+2) = \frac{\langle \chi_{(n_1+m_1),((n_1),(m_1))} \chi_{(n_2+m_2),((n_2),(m_2))} \chi^{\dagger}_{(n_{12}+m_{12}),((n_{12}),(m_{12}))}\rangle}{||\chi_{(n_1+m_1),((n_1),(m_1))}|| \, ||\chi_{(n_2+m_2),((n_2),(m_2))}|| \, ||\chi_{(n_{12}+m_{12}),((n_{12}),(m_{12}))}||}$$

$$= \sqrt{\frac{(N+n_{12}+m_{12}-1)!(N-1)!}{(N+m_1+n_1-1)!(N+n_2+m_2-1)!}} \sqrt{\frac{n_{12}!m_{12}!(n_1+m_1)!(n_2+m_2)!}{n_1!n_2!m_1!m_2!(n_{12}+m_{12})!}}.$$

It is again easy to verify that if we set $m_1 = 0 = m_2$, we recover the one matrix result of [3]. For $\frac{n_1+m_1}{N}$ and $\frac{n_2+m_2}{N}$ fixed and small in the limit $N \to \infty$, we find $\Gamma(1;2|1+2) \sim e^{\alpha N}$ with α positive and $O(1)$ suggesting that these restricted Schur polynomials do not provide weakly coupled degrees of freedom for high energy modes.

5. Interpretation of restricted Schur polynomials

The restricted Schur polynomial $\chi_{R,(r_n,r_m)}(Z,X)$ has three labels R, r_n and r_m. The label r_n is naturally associated to the Zs and the label r_m to the Xs. It seems natural to think that the composite operator $\chi_{R,(r_n,r_m)}(Z,X)$ is constructed from the half BPS "partons" $\chi_{r_n}(Z)$ and $\chi_{r_m}(X)$. In this section we will see if we can gather some evidence that $\chi_{R,(r_n,r_m)}(Z,X)$ indeed has a partonic structure.

Before considering the parton structure of the restricted Schur polynomials, we recall a related relevant problem: the parton structure of hadrons [30]. At high energy, above about $10\,\text{GeV}$, proton-proton scattering produces a large number of pions, with momenta mainly collinear with the collision axis. The probability of producing a pion with a large component of momentum transverse to the collision axis is exponentially suppressed with the value of the transverse momentum. A picture of the proton as a loosely bound assemblage of partons was consistent with the observed data. It is at high energies, where the partons become weakly interacting (thanks to asymptotic freedom) that they are visible in the experimental results. At low energies the constituents are strongly interacting and the parton structure of the proton is not visible.

Is there an analogous limit in which we expect the constituents of the restricted Schur polynomial are weakly interacting? Do we see evidence for a partonic structure in this limit?

There are two distinct ways in which we could obtain a weakly interacting system. The parameter which controls the interactions among our proposed constituents is $\frac{1}{N}$. We

could thus imagine taking $N \to \infty$ and systematically expanding our (exact) correlator results, keeping only the leading order. Alternatively, we could ask if there are situations where we expect the constituents are naturally weakly interacting. We will follow this second approach. Interactions between membranes are mediated by the open strings ending on the membrane's worldvolume; in general the string and the membrane can exchange momentum, that is, the string can exert a force on the membrane. For the special case of a nearly maximal sphere giant,[3] this interaction is highly suppressed and the open strings attached to a maximal giant do not exert a force on it [31]. Thus, to get weakly interacting partons, we consider a bound state built from two partons, one of which is a boundstate of nearly maximal sphere giants. In what follows, we take r_n to be a Young diagram with c columns and p rows with c a number of $O(1)$ and $N - p$ a number of $O(1)$. From the results of section 3.3 we know that

$$\chi_{r_n}(Z)\chi_{r_m}(X) = \frac{n!m!}{(n+m)!d_{r_n}d_{r_m}} \sum_R \sum_i d_R \chi_{R,(r_n^{(i)},r_m^{(i)})}(Z,X),$$

where the sum is over all representations R that can subduce (r_n, r_m) and (i) is a multiplicty label distinguishing the different (r_n, r_m) representations that can be subduced. If we normalize the above operators so that they have a unit two point function we obtain

$$\hat{\chi}_{r_n}(Z)\hat{\chi}_{r_m}(X) = \sum_R \sum_i \sqrt{\frac{\mathrm{Dim}_R}{\mathrm{Dim}_{r_n}\mathrm{Dim}_{r_m}}}\hat{\chi}_{R,(r_n^{(i)},r_m^{(i)})}(Z,X), \tag{5.1}$$

where a hat denotes operators with unit two point function and Dim_R denotes the dimension of the $\mathrm{SU}(N)$ irreducible representation labeled by R. It is simple to check that there is a single Young diagram R that dominates the sum on the right hand side; it has the form displayed in figure 1 below. Indeed, any boxes stacked below r_n must be antisymmetrized with the other indices in the same column of the Young diagram; since the number of boxes already in the column is p, the index corresponding to the boxes stacked below r_n can only take $N - p = O(1)$ values. In addition, one divides by a big symmetry factor (it is $O(N)$). In contrast to this, the indices corresponding to boxes stacked next to r_n are symmetrized with boxes appearing in the same row. These can take $O(N)$ values. Since the number of boxes in the row is $O(1)$ one divides by an extra symmetry factor of $O(1)$. There is a single way to subduce this dominant R so that there is no need for the i index. It is now clear that the leading correction to this term is of order $\frac{1}{N}$. Lets now consider the interaction of $\hat{\chi}_{R,(r_{n_1},r_{m_1})}(Z,X)$ and $\hat{\chi}_{S,(s_{n_2},s_{m_2})}(Z,X)$, where both r_{n_1} and s_{n_2} are boundstates of nearly maximal sphere giants. This is the analog of scattering two protons. Using (5.1) it is now straight forward to see that

$$\chi_{R,(r_{n_1},r_{m_1})}(Z,X)\chi_{S,(s_{n_2},s_{m_2})}(Z,X) = \frac{(\mathrm{hooks})_R(\mathrm{hooks})_S}{(\mathrm{hooks})_{r_{n_1}}(\mathrm{hooks})_{r_{m_1}}(\mathrm{hooks})_{s_{n_2}}(\mathrm{hooks})_{s_{m_2}}}$$

$$\times \sum_{t_{n_{12}}} \sum_{t_{m_{12}}} \sum_T \sum_i \frac{n_{12}!m_{12}!d_T}{(n_{12}+m_{12})!d_{t_{n_{12}}}d_{t_{m_{12}}}} f_{r_{n_1}s_{n_2}t_{n_{12}}} f_{r_{m_1}s_{m_2}t_{m_{12}}} \chi_{T,(t_{n_{12}}^{(i)},t_{m_{12}}^{(i)})}(Z,X).$$

---

[3]To be more precise, we consider a giant graviton which carries momentum p with $N - p$ a number of $O(1)$.
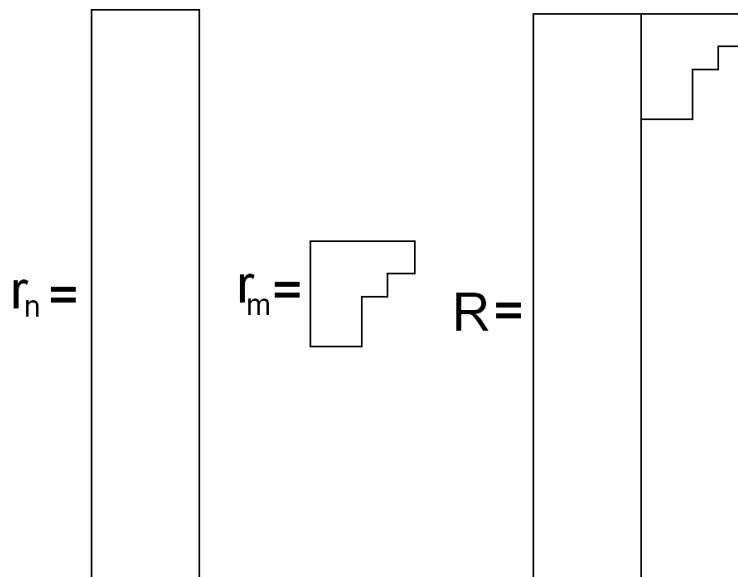
Figure 1: The Young diagram which dominates (5.1) is obtained by stacking r_m next to r_n as shown.

Even in the large N limit, the right hand side is a sum over a number of terms - many possible states can be formed as a result of the interaction of our two composites. Note however that the detailed structure of this interaction, in particular which representations $t_{n_{12}}$ and $t_{m_{12}}$ appear, is determined by the Littlewood-Richardson coefficients $f_{r_{n_1} s_{n_2} t_{n_1+n_2}}$ and $f_{r_{m_1} s_{m_2} t_{m_1+m_2}}$. These Littlewood-Richardson coefficients give the detailed form of the interaction between the $\frac{1}{2}$ BPS partons

$$\chi_{r_{n_1}}(Z)\chi_{s_{n_2}}(Z) = \sum_{t_{n_1+n_2}} f_{r_{n_1} s_{n_2} t_{n_1+n_2}} \chi_{t_{n_1+n_2}}(Z),$$

$$\chi_{r_{m_1}}(X)\chi_{s_{m_2}}(X) = \sum_{t_{m_1+m_2}} f_{r_{m_1} s_{m_2} t_{m_1+m_2}} \chi_{t_{m_1+m_2}}(X).$$

Thus, the interaction of the composites is as a result of interactions between the partons. The picture that has emerged is very similar to the parton structure of hadrons described above.

6. Discussion

In this article we have demonstrated that the restricted Schur polynomials satisfy a simple product rule, generalizing the rule known for Schur polynomials. Using this rule it is now possible to perform computations of higher point correlation functions of restricted Schur polynomials. We have obtained explicit formulas for three point functions of a large class of restricted Schur polynomials which are built using antisymmetric or symmetric representations. Using these results we have been able to argue that restricted Schur polynomials built using the antisymmetric representations provide a suitable set of degrees

of freedom for the description of perturbative quantum gravity, in agreement with the conclusions of [22] and extending the explicit computations performed there.

A key building block appearing in the restricted Schur polynomial is the restricted character. The restricted character is a generalization of the usual character, which plays a central role in group theory. In this paper there is a development of the restricted character theory parallel to the character theory of finite groups. The notions of a conjugacy class and of a dual character, have been generalized to the notions of restricted conjugacy class and dual restricted characters. These generalizations lead to a set of orthogonality relations satisfied by the restricted characters. In this way we have ultimately obtained the generalization of the Littlewood-Richardson coefficient.

By studying the interaction of two restricted Schur polynomials we have suggested a physical interpretation for the labels of the restricted Schur polynomial: the composite operator $\chi_{R,(r_n,r_m)}(Z,X)$ is constructed from the half BPS "partons" $\chi_{r_n}(Z)$ and $\chi_{r_m}(X)$. Specifically, we have identified a composite operator in which the constituent partons are weakly interacting. The interaction of two such composites is largely determined by the interactions of the partons. This is analogous to the shower of pions produced when two hadrons interact; the pion shower can be attributed to the parton-parton interactions between pairs of partons belonging to different hadrons. In cases when the partons are strongly interacting (for example, at low energy), they are not visible and a partonic description of hadrons is not useful. In a similar way, we expect that when the constituent half BPS partons are strongly interacting, this might not be a useful interpretation of the restricted Schur polynomials. The picture of a composite comprised of half BPS partons seems to be similar to the proposal of [32] that every supersymmetric four dimensional black hole of finite area can be split up into microstates made of primitive half-BPS "atoms". The idea that each half BPS state should be treated as an independent parton matches nicely with the picture that has emerged for half-BPS states in AdS$_5$ [2, 33]. Finally, it is not yet known how to build restricted Schur polynomials that correspond to $\frac{1}{4}$ or $\frac{1}{8}$ BPS states. In [34] $\frac{1}{8}$ and $\frac{1}{4}$ BPS states were obtained by putting together $\frac{1}{2}$ BPS dual giants; making contact with that work may indicate which restricted Schur polynomials correspond to $\frac{1}{4}$ or $\frac{1}{8}$ BPS states.

Although we have focused on the case of two matrices, the extension to more matrices is straight forward as described in appendix C. It would be interesting to work out some explicit examples with more than two matrices. Further, even for two matrices, it would be nice to make the formalism developed here more efficient. Indeed, in computing the values of two cycles with one index in the S_n subgroup and one index in the S_m subgroup, we have built a change of basis. This allows us not only to compute the value of the trace, but the value of any matrix element. This is much more information than we actually need; surely more efficient methods can be found.

The Littlewood-Richardson numbers have many interpretations: as coefficients in the decomposition of tensor products into irreducible GL_n modules, as coefficients in the decomposition of skew Specht modules into irreducibles, as coefficients in the decomposition of S_n representations induced from Young subgroups and as intersection numbers in the Schubert calculus on a Grassmannian [35]. How much of this generalizes for restricted

Littlewood-Richardson numbers? Finally, our work contains some general directions for the study of multi-matrix models. The eigenvalue based techniques that were so useful for the description of one matrix models do not seem to have an easy generalization for multi matrices. For one matrix models, the eigenvalues provide a set of $O(N)$ variables; since fluctuations about the large N configuration are $O(N^{-2})$, these provide a suitable set of variables for the implementation of a saddle point approximation. In general, it would be incorrect to assume that the matrices of a multimatrix model can be simultaneously diagonalized.[4] Thus, at best it seems that one needs to keep the eigenvalues of each matrix, plus unitary matrices which encode how one goes between the bases in which a particular matrix is diagonal. This gives a total of $O(N^2)$ variables, so that this does not provide a promising starting point for a saddle point analysis. In contrast to this, using the technology of [14, 9 – 11], we can give a rather detailed description of the free multi-matrix models. One loop results for the super Yang-Mills F term also have a description in this framework [15, 16, 19]. It would be interesting to develop these methods further by developing efficient techniques for extracting large N results and for managing more general potentials.

Acknowledgments

A. Identities and notation

The polynomials we construct are built out of two matrices, X and Z. We will typically use an n to denote the number of Zs in the polynomial and an m to denote the number of Xs in the polynomial. The polynomial is built using an irreducible representation of the symmetric group S_{n+m} that permutes the indices of the Z and X fields amongst each other. We will denote the Young diagrams labeling representations of the symmetric group acting on both Xs and Zs using capital letters. The indices of the matrix representing an element of this group will be denoted by capital Greek letters. Thus, the elements of the matrix representing σ in the S_{n+m} irreducible representation R will be denoted by $\left[\Gamma_R(\sigma)\right]_{\Psi\Phi}$. The irreducible representations of the S_n subgroup that acts only on the indices of the Z fields will be denoted by a small letter with a subscript n; matrix elements of this representation will be indexed using letters from the start of the alphabet. Thus, the elements of the matrix representing σ in the S_n irreducible representation r_n will be denoted by $\left[\Gamma_{r_n}(\sigma)\right]_{ab}$. The irreducible representations of the S_m subgroup that acts only on the indices of the X fields will be denoted by a small letter with a subscript m; matrix elements

---

[4]An exception to this is the sector of the theory comprising of the chiral ring at strong coupling - see [36].

of this representation will be indexed using letters from the middle of the alphabet. Thus, the elements of the matrix representing σ in the S_m irreducible representation r_m will be denoted by $\left[\Gamma_{r_m}(\sigma)\right]_{ij}$. Finally, irreducible representations of the $S_n \times S_m$ subgroup that permute the indices of the Xs and permute the indices of the Zs is denoted by (r_n, r_m).

The delta function $\delta(\sigma)$ is 1 if σ is the identity and zero otherwise. In this article the $\delta(\cdot)$ is usually summed with a summand that is a class function. In addition, we usually have a delta function defined on the classes, i.e. we have $\delta\left(\left[\sigma\right]_r \left[\tau\right]_r\right)$ instead of $\delta(\sigma\tau)$. The relation between these two is easily established: Let $F(\left[\sigma\right]_r)$ be any class function. By comparing

$$\sum_{\sigma \in S_n} F(\left[\sigma\right]_r)\delta(\left[\sigma\right]_r \left[\tau\right]_r) \qquad \text{and} \qquad \sum_{\sigma \in S_n} F(\left[\sigma\right]_r)\delta(\sigma\tau)$$

we find that, when summing the delta function multiplied by any class function, over the whole group, we can freely replace

$$\frac{\delta(\left[\sigma\right]_r \left[\tau\right]_r)}{n^{\sigma}_{R,(r_n,r_m)}} \leftrightarrow \delta(\sigma\tau).$$

B. A formula for the restricted Littlewood-Richardson numbers

In this appendix we will employ a bra/ket notation. In what follows, R is an irreducible representation of S_{n+m} and (r_n, r_m) is an irreducible representation of $S_n \times S_m$. We label the irreducible representations of $S_n \times S_m$ by a Young diagram r_n with n boxes (which labels an irreducible representation of S_n) and a Young diagram r_m with m boxes (which labels an irreducible representation of S_m). For states belonging to the carrier space of R we write $|R, \Gamma\rangle$. For states belonging to the carrier space of (r_n, r_m) we write labels acted on by the S_n and S_m subgroups separately $|r_n, a; r_m, i\rangle$. For example, we write

$$\left[\Gamma_R(\alpha)\right]_{\Lambda\Psi} = \langle R, \Lambda|\alpha|R, \Psi\rangle.$$

For a trace over an "on the diagonal block" (see [13, 14] for an explanation of this terminology) we write

$$\text{Tr}_{(r_n,r_m)}(\Gamma_R(\sigma)) = \sum_{i,a} \sum_{\Lambda\Theta} \langle r_n, a; r_m, i|R, \Lambda\rangle \left[\Gamma_R(\sigma)\right]_{\Lambda\Theta} \langle R, \Theta|r_n, a; r_m, i\rangle.$$

For an "off the diagonal block" we write

$$\text{Tr}_{(r_n,r_m),(s_n,s_m)}(\Gamma_R(\sigma)) = \sum_{i,a} \sum_{\Lambda\Theta} \langle r_n, a; r_m, i|R, \Lambda\rangle \left[\Gamma_R(\sigma)\right]_{\Lambda\Theta} \langle R, \Theta|s_n, a; s_m, i\rangle.$$

For the labels of the off the diagonal block, we need (r_n, r_m) to have the same shape as (s_n, s_m). This means that r_n and s_n as well as s_m and r_m have the same shape.

For $\alpha_1 \in S_n$ and $\alpha_2 \in S_m$ we have

$$
\begin{aligned}
\left[\Gamma_R(\alpha_1 \circ \alpha_2)\right]_{\Psi\Theta} &= \langle R, \Psi|\alpha_1 \circ \alpha_2|R, \Theta\rangle \\
&= \sum_{r_n,r_m,a,i} \sum_{t_n,t_m,b,j} \langle R, \Psi|r_n, a; r_m, i\rangle\langle r_n, a; r_m, i|\alpha_1 \circ \alpha_2|t_n, b; t_m, j\rangle \\
&\qquad \times \langle t_n, b; t_m, j|R, \Theta\rangle
\end{aligned}
$$

$$= \sum_{r_n, r_m, a, i} \sum_{b, j} \langle R, \Psi | r_n, a; r_m, i \rangle \big[\Gamma_{r_n}(\alpha_1)\big]_{ab} \big[\Gamma_{r_m}(\alpha_2)\big]_{ij}$$
$$\times \langle r_n, b; r_m, j | R, \Theta \rangle .$$

To get to the last line we have used the fact that if $\alpha_1 \in S_n$ and $\alpha_2 \in S_m$ then $\langle r_n, a; r_m, i | \alpha_1 \circ \alpha_2 | t_n, b; t_m, j \rangle \propto \delta_{r_n t_n} \delta_{r_m t_m}$. Use this identity to compute

$$\sum_{\alpha_1 \in S_n} \sum_{\alpha_2 \in S_m} \big[\Gamma_R(\alpha_1 \circ \alpha_2)\big]_{\Psi\Theta} \big[\Gamma_{r_n}(\alpha_1^{-1})\big]_{ab} \big[\Gamma_{r_m}(\alpha_2^{-1})\big]_{ij}$$

$$= \sum_{\alpha_1, \alpha_2} \sum_{t_n t_m} \sum_{c d k l} \langle R, \Psi | t_n, c; t_m, k \rangle \big[\Gamma_{t_n}(\alpha_1)\big]_{cd} \big[\Gamma_{t_m}(\alpha_2)\big]_{kl}$$
$$\times \langle t_n, d; t_m, l | R, \Theta \rangle \big[\Gamma_{r_n}(\alpha_1^{-1})\big]_{ab} \big[\Gamma_{r_m}(\alpha_2^{-1})\big]_{ij}$$

$$= \frac{n! m!}{d_{r_n} d_{r_m}} \sum_i \langle R, \Psi | r_n^{(i)}, b; r_m^{(i)}, j \rangle \langle r_n^{(i)}, a; r_m^{(i)}, i | R, \Theta \rangle .$$

To obtain this result we have used the fundamental orthogonality relation

$$\sum_{\alpha \in \mathcal{G}} \big[\Gamma_R(\alpha)\big]_{ab} \big[\Gamma_S(\alpha^{-1})\big]_{cd} = \frac{g}{d_R} \delta_{\mathrm{RS}} \delta_{ad} \delta_{bc},$$

with d_R the dimension of irreducible representation R and g the order of $\mathcal{G}$. A given $S_n \times S_m$ irreducible representation may be subduced more than once by an S_{n+m} irreducible representation R. When applying this relation to the sums over α_1 and α_2, we will get a contribution from all representations whose S_n and S_m labels match. The index i in $(r_n^{(i)}, r_m^{(i)})$ runs over the complete set of identical $S_n \times S_m$ irreducible representations. This last identity will be used to argue that the dual character is

$$\chi^{R, R_\gamma}(\tau) = \chi^{R, (r_n^{(i)}, r_m^{(i)})(r_n^{(j)}, r_m^{(j)})}(\tau) = \frac{d_R n! m!}{d_{r_n} d_{r_m} (n+m)!} \chi_{R, (r_n^{(j)}, r_m^{(j)})(r_n^{(i)}, r_m^{(i)})}(\tau).$$

To verify this, we compute

$$\sum_{R, R_\gamma} \chi^{R, R_\gamma}(\tau) \chi_{R, R_\gamma}(\sigma)$$

$$= \sum_R \sum_{r_n^{(i)}, r_m^{(i)}} \sum_{r_n^{(j)}, r_m^{(j)}} \chi^{R, (r_n^{(i)}, r_m^{(i)})(r_n^{(j)}, r_m^{(j)})}(\tau) \chi_{R, (r_n^{(i)}, r_m^{(i)})(r_n^{(j)}, r_m^{(j)})}(\sigma)$$

$$= \sum_R \sum_{r_n^{(i)}, r_m^{(i)}} \sum_{r_n^{(j)}, r_m^{(j)}} \sum_{\Theta, \Psi, \Gamma, \Sigma} \sum_{a, b, i, j}$$
$$\Gamma_R(\sigma)_{\Theta\Psi} \Gamma_R(\tau)_{\Gamma\Sigma} \langle r_n^{(i)}, a; r_m^{(i)}, i | R, \Theta \rangle \langle R, \Psi | r_n^{(j)}, a; r_m^{(j)}, i \rangle$$
$$\times \langle r_n^{(j)}, b; r_m^{(j)}, j | R, \Gamma \rangle \langle R, \Sigma | r_n^{(i)}, b; r_m^{(i)}, j \rangle \frac{d_R n! m!}{d_{r_n} d_{r_m} (n+m)!}$$

$$= \sum_R \sum_{r_n, r_m} \sum_{\Theta, \Psi, \Gamma, \Sigma} \sum_{a, b, i, j} \Gamma_R(\sigma)_{\Theta\Psi} \Gamma_R(\tau)_{\Gamma\Sigma} \frac{d_R n! m!}{d_{r_n} d_{r_m} (n+m)!} \left(\frac{d_{r_n} d_{r_m}}{n! m!}\right)^2$$
$$\times \sum_{\alpha_1, \alpha_2} \big[\Gamma_R(\alpha_1 \circ \alpha_2)\big]_{\Sigma\Theta} \big[\Gamma_{r_n}(\alpha_1^{-1})\big]_{ba} \big[\Gamma_{r_m}(\alpha_2^{-1})\big]_{ji}$$

$$\times \sum_{\beta_1,\beta_2} \left[\Gamma_R(\beta_1 \circ \beta_2)\right]_{\Psi\Gamma} \left[\Gamma_{r_n}(\beta_1^{-1})\right]_{ab} \left[\Gamma_{r_m}(\beta_2^{-1})\right]_{ij}$$

$$= \sum_R \sum_{\alpha_1,\alpha_2} \Gamma_R(\sigma)_{\Theta\Psi} \Gamma_R(\tau)_{\Gamma\Sigma} \frac{d_R}{(n+m)!} \left[\Gamma_R(\alpha_1 \circ \alpha_2)\right]_{\Sigma\Theta} \left[\Gamma_R(\alpha_1^{-1} \circ \alpha_2^{-1})\right]_{\Psi\Gamma}$$

$$= \sum_R \sum_{\alpha_1,\alpha_2} \chi_R(\sigma(\alpha_1^{-1} \circ \alpha_2^{-1})\tau(\alpha_1 \circ \alpha_2)) \frac{d_R}{(n+m)!}$$

$$= n!m!\delta([\sigma]_r[\tau]_r),$$

which demonstrates the result. In the above proof we have made use of the formula

$$\sum_R \frac{d_R}{g} \chi_R(\sigma) = \delta(\sigma).$$

Clearly then,

$$f_{R,R_\alpha\,S,S_\beta}^{T,(t_n^{(i)},t_m^{(i)})(t_n^{(j)},t_m^{(j)})} = \frac{1}{n_1!n_2!m_1!m_2!} \sum_{\sigma_1 \in S_{n_1+m_1}} \sum_{\sigma_2 \in S_{n_2+m_2}}$$

$$\frac{d_T(n_1+n_2)!(m_1+m_2)!}{d_{t_n}d_{t_m}(n_1+n_2+m_1+m_2)!} \chi_{R,R_\alpha}(\sigma_1)\chi_{S,S_\beta}(\sigma_2)\chi_{T,(t_n^{(i)},t_m^{(j)})(t_n^{(i)},t_m^{(i)})}(\sigma_1 \circ \sigma_2).$$

C. More than 2 matrices

Consider a matrix model with d species of matrices. R is an irreducible representation of $S_{n_1+n_2+\cdots+n_d}$ and $(r_{n_1}, r_{n_2}, \ldots, r_{n_d})$ is an irreducible representation of $S_{n_1} \times S_{n_2} \times \cdots \times S_{n_d}$. The Young diagram r_{n_i} has n_i boxes; it labels an irreducible representation of S_{n_i}. It is straight forward to show that

$$\prod_{i=1}^d \sum_{\alpha_i \in S_{n_i}} \left[\Gamma_R(\alpha_1 \circ \alpha_2 \circ \cdots \circ \alpha_d)\right]_{\Psi\Theta} \left[\Gamma_{r_{n_1}}(\alpha_1^{-1})\right]_{a_1b_1} \cdots \left[\Gamma_{r_{n_d}}(\alpha_d^{-1})\right]_{a_db_d}$$

$$= \prod_{i=1}^d \frac{n_i!}{d_{r_{n_i}}} \sum_j \langle R, \Psi | r_{n_1}^{(j)}, b_1; r_{n_2}^{(j)}, b_2; \ldots r_{n_d}^{(j)}, b_d \rangle \langle r_{n_1}^{(j)}, a_1; r_{n_2}^{(j)} a_2; \ldots ; r_{n_d}^{(j)} a_d | R, \Theta \rangle.$$

This last identity can again be used to argue that the dual character is

$$\chi^{R,(r_{n_1}^{(i)},r_{n_2}^{(i)},\ldots,r_{n_d}^{(i)})(r_{n_1}^{(j)},r_{n_2}^{(j)},\ldots,r_{n_d}^{(j)})}(\tau) =$$

$$\prod_{k=1}^d \frac{n_k!}{d_{r_k}} \frac{d_R}{(n_1+n_2+\cdots+n_k)!} \chi_{R,(r_{n_1}^{(j)},r_{n_2}^{(j)},\ldots,r_{n_d}^{(j)})(r_{n_1}^{(i)},r_{n_2}^{(i)},\ldots,r_{n_d}^{(i)})}(\tau).$$

Clearly then, (R is an irreducible representation of $S_{n_1+n_2+\cdots+n_d}$; R_α is an irreducible representation of $S_{n_1} \times S_{n_2} \times \cdots \times S_{n_d}$; S is an irreducible representation of $S_{m_1+m_2+\cdots+m_d}$; S_β is an irreducible representation of $S_{m_1} \times S_{m_2} \times \cdots \times S_{m_d}$; T is an irreducible representation of $S_{n_1+n_2+\cdots+n_d+m_1+m_2+\cdots+m_d}$; $(r_{n_1}^{(i)}, r_{n_2}^{(i)}, \ldots, r_{n_d}^{(i)})$ and $(r_{n_1}^{(j)}, r_{n_2}^{(j)}, \ldots, r_{n_d}^{(j)})$ have the same shape and are both irreducible representations of $S_{n_1+m_1} \times S_{n_2+m_2} \times \cdots \times S_{n_d+m_d}$)

$$f_{R,R_\alpha\,S,S_\beta}^{T,(r_{n_1}^{(i)},r_{n_2}^{(i)},\ldots,r_{n_d}^{(i)})(r_{n_1}^{(j)},r_{n_2}^{(j)},\ldots,r_{n_d}^{(j)})} = \prod_{i=1}^d \frac{1}{n_i!m_i!} \sum_{\sigma_1 \in S_{n_1+n_2+\cdots+n_d}} \sum_{\sigma_2 \in S_{m_1+m_2+\cdots+m_d}}$$

$$\prod_{k=1}^{d} \frac{(n_k + m_k)!}{d_{r_k}} \frac{d_R}{(n_1 + n_2 + \cdots + n_k + m_1 + m_2 + \cdots + m_k)!} \times$$

$$\times \chi_{R,R_\alpha}(\sigma_1) \chi_{S,S_\beta}(\sigma_2) \chi_{T,(r_{n_1}^{(j)}, r_{n_2}^{(j)}, \ldots, r_{n_d}^{(j)})(r_{n_1}^{(i)}, r_{n_2}^{(i)}, \ldots, r_{n_d}^{(i)})}(\tau).$$

D. Restricted character computations

In this appendix we will consider the computation of the restricted character, for arbitrary representations, on the $m + n$ cycle $\sigma_c = (1, 2, 3, \ldots, m + n)$. This character is needed to evaluate the two point functions used in section 4. Representations are labeled by a Young diagram. To specify a Young diagram, we will list the number of boxes in each row. In this appendix, the Young diagrams, called "hooks" in [27] will play an important role. Listing the number of boxes in each row, the hook diagrams are always of the form $(n + m - s, 1, 1, \ldots, 1)$. Recall that the symmetric group character $\chi_R(\sigma_c)$ is $(-1)^s$ if R is a hook and zero otherwise. Let us verify this formula, using strand diagrams, for the hook ⊞. There are two possible ways of removing the three boxes

$$\boxed{\begin{array}{cc}3&2\\1\end{array}} \quad \text{or} \quad \boxed{\begin{array}{cc}3&1\\2\end{array}}.$$

Using the decomposition

$$(123) = (12)(23),$$

the strand diagram gives

$$\frac{1}{c_1 - c_2} \frac{1}{c_2 - c_3} = -\frac{1}{2} \times 1 \quad \text{for} \quad \boxed{\begin{array}{cc}3&2\\1\end{array}}$$

$$= \frac{1}{2} \times -1 \quad \text{for} \quad \boxed{\begin{array}{cc}3&1\\2\end{array}}. \tag{D.1}$$

The sum of these contributions is -1 as it should be. Refer to a particular order of removing the boxes from the Young diagram as a *path* through the Young diagram. The strand diagram computation for the character of the $m + n$ cycle σ_c in the hook representation $(n + m - s, 1, 1, \ldots, 1)$ implies that

$$\sum_{\text{paths}} \prod_{i=1}^{m+n-1} \frac{1}{c_i - c_{i+1}} = (-1)^s.$$

First we establish that the restricted character of the $m + n$ cycle σ_c with representation $(n + m - s, 1, 1, \ldots, 1)$, and representation of the restriction obtained either by removing the box from the last or the first row. In this case, don't sum all paths - only sum paths

that start from the last or first row respectively. Denote these two sums graphically as

$$\text{[hook diagram, 1 in top-right]} = \sum_{\text{paths starting in row }1} \prod_{i=1}^{m+n-1} \frac{1}{c_i - c_{i+1}},$$

$$\text{[hook diagram, 1 in bottom-left]} = \sum_{\text{paths starting in row }s+1} \prod_{i=1}^{m+n-1} \frac{1}{c_i - c_{i+1}}.$$

The result we will establish says that (recall that there are $m + n$ boxes in the hook)

$$\text{[hook diagram, 1 in top-right]} = (-1)^s \frac{m+n-1-s}{m+n-1},$$

$$\text{[hook diagram, 1 in bottom-left]} = (-1)^s \frac{s}{m+n-1}.$$

Assuming this result is true for a hook with $m + n$ boxes and $s + 1$ rows, it is straight forward to prove it is true for a hook with $m + n + 1$ boxes and $s + 1$ or $s + 2$ rows. Indeed, for $m + n + 1$ boxes in the hook and $m + n + 1 - s$ boxes in the first row (so that the hook has $s + 1$ rows), we have

$$
\text{[hook, 1 top-right]} = \text{[hook, 2|1]} + \text{[hook, 1 top-right, 2 below]}
$$

$$
= \text{[hook, 1 top-right]} + \frac{1}{m+n}\text{[hook, 1]}
$$

$$
= \frac{m+n-s-1}{m+n-1}(-1)^s + \frac{1}{m+n}\frac{s}{m+n-1}(-1)^s
$$

$$
= \frac{m+n-s}{m+n}(-1)^s .
$$

For $m + n + 1$ boxes in the hook and $m + n - s$ boxes in the first row (so that the hook has $s + 2$ rows), we have

$$
\text{[hook, 1 bottom]} = \text{[hook, 2|1 bottom]} + \text{[hook, 2,1 diagram]}
$$

$$
= -\frac{1}{m+n}\text{[hook, 1]} - \text{[hook, 1]}
$$

$$= -\frac{1}{m+n}\frac{m+n-s-1}{m+n-1}(-1)^s - \frac{s}{m+n-1}(-1)^s$$
$$= \frac{s+1}{m+n}(-1)^{s+1}.$$

Thus, to establish the result it is enough to show that

$$\boxed{\begin{array}{c}1\end{array}} = -\frac{1}{2} = \boxed{1},$$

which has already been demonstrated in (D.1) above.

We are now ready to tackle the computation of $\mathrm{Tr}_{(r_n,r_m)}(\Gamma_R(\sigma_c))$. We again use a strand diagram technique, which amounts to summing over all paths and decomposing σ_c into a product of two cycles. The values of all two cycles $(i,i+1)$ *except for* $(n,n+1)$ are given by $(c_i - c_{i+1})^{-1}$ with the weights c_i read off the paths, coming from r_n for $i \leq n-1$ or from r_m for $i > n$. If we strip off the boxes belonging to r_n first, we can factor out a term which equals the character of $(1, 2, \ldots, n)$ in irreducible representation r_n so that $\mathrm{Tr}_{(r_n,r_m)}(\Gamma_R(\sigma_c))$ vanishes unless r_n is a hook. Stripping off the boxes that belong to r_m first, allows us to conclude that $\mathrm{Tr}_{(r_n,r_m)}(\Gamma_R(\sigma_c))$ again vanishes unless r_m is a hook. Let the hook associated with r_n have n boxes, arranged as $(n - s_n, 1, 1, \ldots, 1)$ and let the hook associated with r_m have m boxes, arranged as $(m - s_m, 1, 1, \ldots, 1)$. Given that both r_n and r_m have to be hooks, what are the allowed values of R? Further, given these values, what is $\mathrm{Tr}_{(r_n,r_m)}(\Gamma_R(\sigma_c))$? The allowed values of R fit into four possible cases. In what follows we will list the structure of the $R, (r_n, r_m)$ label for these four cases.

Case 1: r_m contained in r_n with no overlap. To denote the structure of this case we display R together with an x in the boxes which are removed to give r_m
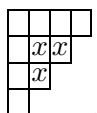
In general, the last box removed from r_m does not correspond to a specific box in R. In this case the number $c_n - c_{n+1}$ appearing in the usual strand diagram computations is not well defined - the strand diagrams techniques of [16] are not applicable. However, for case 1, the box in the second row and second column is the last box of r_m that is removed. Thus, $c_n - c_{n+1}$ has a definite value and it is simple to compute the value of the $(n, n+1)$ cycle. Using the formulas given above, it is straight forward to verify that

$$\mathrm{Tr}_{(r_n,r_m)}(\Gamma_R(\sigma_c)) = -(-1)^{s_m}\frac{1}{s_n}\frac{s_n(-1)^{s_n}}{n-1} + (-1)^{s_m}\frac{1}{n-s_n-1}\frac{(n-s_n-1)(-1)^{s_n}}{n-1}$$
$$= 0.$$

Case 2: r_m contained in r_n with row overlap. Display R together with an x in the boxes which are removed to give r_m
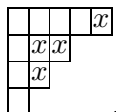
In this case, the box in the second row and second column or the last marked box in the first row, is the last box of r_m that is removed. The argument for case 1 does not easily generalize to case 2 (or cases 3 and 4).

Case 3: r_m contained in r_n with column overlap. Display R together with an x in the boxes which are removed to give r_m
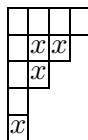
Case 4: r_m contained in r_n with row and column overlap. Display R together with an x in the boxes which are removed to give r_m
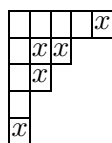
We will now give an argument applicable to all four cases. As a nontrivial check of our general formula, we will verify that it predicts zero for case 1. Decompose our $n + m$ cycle σ_c as

$$\sigma_c = (1, 2, \ldots, n)(n, n + 1)(n + 1, n + 2, \ldots, n + m).$$

Using the $S_n \times S_m$ symmetry enjoyed by our restricted character, we can replace

$$(1, 2, \ldots, n) \to \mathcal{C}_n = \frac{\sum_{\text{n cycles}}(i_1, i_2, \ldots, i_n)}{(n - 1)!},$$

$$(n + 1, n + 2, \ldots, n + m) \to \mathcal{C}_m = \frac{\sum_{\text{m cycles}}(i_1, i_2, \ldots, i_m)}{(m - 1)!}.$$

$\mathcal{C}_n$ is a sum over the $(n - 1)!$ n-cycles in S_n; $\mathcal{C}_m$ is a sum over the $(m - 1)!$ m-cycles in S_m. It is clear that $\mathcal{C}_n$ and $\mathcal{C}_m$ are Casimirs of the (r_n, r_m) representation. Using the known characters of the hooks, it is straight forward to see that

$$\text{Tr}\,(\mathcal{C}_n) = (-1)^{s_n}, \qquad \text{Tr}\,(\mathcal{C}_m) = (-1)^{s_m},$$

so that these two Casimirs have eigenvalue

$$\mathcal{C}_n = \frac{(-1)^{s_n}}{d_{r_n}}, \qquad \mathcal{C}_m = \frac{(-1)^{s_m}}{d_{r_m}}.$$

Thus, we now have

$$\begin{aligned}
\text{Tr}\,_{(r_n, r_m)}\left(\Gamma_R(\sigma_c)\right) &= \text{Tr}\,_{(r_n, r_m)}\left(\mathcal{C}_n(n, n + 1)\mathcal{C}_m\right) \\
&= \frac{(-1)^{s_n}}{d_{r_n}}\frac{(-1)^{s_m}}{d_{r_m}}\text{Tr}\,_{(r_n, r_m)}\left((n, n + 1)\right) \\
&= \frac{(-1)^{s_n + s_m}}{mn}\left(\sum_i c_i - mN - \frac{m(m - 2s_m - 1)}{2}\right)
\end{aligned}$$

where to get to the last line we have used formula (3.1). The sum in the last line is over the weights in R which are peeled off and recombined to produce r_m. Notice that to evaluate these hooks we have not needed off the diagonal block traces of the $(n, n + 1)$ character, which is why we are able to obtain a simple and general formula. In addition, this formula is in perfect agreement with the result obtained above for case 1.

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